1. Let \( f(\theta_y) \, d\theta_y \) be the probability per unit length that the particle is scattered through an angle \( \theta_y \) in the \((yz)\) plane.

\[
\langle \theta_y \rangle = \int_{-\infty}^{\infty} \theta_y f(\theta_y) \, d\theta_y = 0 \quad \text{because} \quad f(\theta_y) = f(-\theta_y)
\]

\[
\langle \theta_y^2 \rangle = \int_{-\infty}^{\infty} \theta_y^2 f(\theta_y) \, d\theta_y = \theta_0^2 \quad \text{by definition}
\]

Consider the change in \( N \) going from \( z \to z + dz \)

\[
N(z+dz, y, \theta_y) = N(z, y, \theta_y \, dz, \theta_y) + \sum \left[ N(z, y, \theta_y - \theta_y') f(\theta_y') \, d\theta_y' \, dz + \right]
\]

\[
- \sum N(z, y, \theta_y) f(\theta_y') \, d\theta_y' \, dz
\]

where

- the 1st term is due to the drift, i.e. particles whose displacement changed because \( \theta_y \neq 0 \) (in general)

- the 2nd term represents particles that had the "right" \( y \) but the "wrong" \( \theta_y \) and then scattered into the "right" \( \theta_y \) bin

- the 3rd term represents particles that had
the "right" y and \( \theta_y \), but, due to a further scatter, ended up having the "wrong" \( \theta_y \).

Now let us expand in Taylor series, keeping only leading non-zero terms. Assume that in 2nd and 3rd term contributions are important only when \( \theta_y \) is small. Then

\[
N(z, y, \theta_y) + \frac{\partial N}{\partial z} \, dz = N(z, y, \theta_y) - \frac{\partial N}{\partial y} \, \theta_y \, dz + \\
+ \int N(z, y, \theta_y) \, f(\theta_y') \, d\theta_y' \, dz + \\
- \int \frac{\partial N}{\partial y} \, \theta_y' \, f(\theta_y') \, d\theta_y' \, dz + \\
+ \frac{1}{2} \frac{\partial^2 N}{\partial y^2} \, \theta_y^2 \, f(\theta_y') \, d\theta_y' \, dz + \\
- \int N(z, y, \theta_y) \, f(\theta_y') \, d\theta_y' \, dz
\]

\[
\frac{\partial N}{\partial z} = - \frac{\partial N}{\partial y} \, \theta_y - \frac{\partial N}{\partial y} \int \theta_y' \, f(\theta_y') \, d\theta_y' + \frac{1}{2} \frac{\partial^2 N}{\partial y^2} \int \theta_y'^2 \, f(\theta_y') \, d\theta_y' \\
= <\theta_y> = 0 \\
= <\theta_y^2> = \theta_0^2
\]

\[
\frac{\partial N}{\partial z} = - \theta_y \frac{\partial N}{\partial y} + \frac{1}{2} \theta_0^2 \frac{\partial^2 N}{\partial \theta_y^2}
\]
(ii) Let \[
\begin{align*}
\frac{\partial N}{\partial z} &= \frac{8\sqrt{3}N_0}{\pi \Theta_0^2} \frac{1}{z^3} \exp[\ldots] + \frac{4\sqrt{3}N_0}{\pi \Theta_0^2} \frac{1}{z^2} \left( -\frac{2}{\Theta_0^2} \right) \left( -\frac{2}{\Theta_0^2} + \frac{6y \Theta_0^2}{z^2} - \frac{9y^2}{z^4} \right) \\
\frac{\partial N}{\partial y} &= \frac{8\sqrt{3}N_0}{\pi \Theta_0^2} \frac{1}{z^2} \left( -\frac{2}{\Theta_0^2} \right) \left( -\frac{2}{\Theta_0^2} + \frac{6y \Theta_0^2}{z^2} \right) \exp[\ldots] \\
\frac{\partial N}{\partial \theta_y} &= \frac{8\sqrt{3}N_0}{\pi \Theta_0^2} \frac{1}{z^2} \left( -\frac{2}{\Theta_0^2} \right) \left( -\frac{2}{\Theta_0^2} + \frac{6y \Theta_0^2}{z^2} \right) \exp[\ldots] \\
\frac{\partial N}{\partial \theta_y} &= \frac{8\sqrt{3}N_0}{\pi \Theta_0^2} \left[ -\frac{2}{\Theta_0^2} + \frac{3y}{z^3 \Theta_0^2} \right] \exp[\ldots] \\
\frac{\partial^2 N}{\partial \theta_y^2} &= \frac{8\sqrt{3}N_0}{\pi \Theta_0^2} \left[ -\frac{2}{z^3 \Theta_0^2} + \frac{6y}{\Theta_0^2 z^4} - \frac{12 \theta_y y}{\Theta_0^2 z^5} - \frac{12 \theta_y y}{\Theta_0^2 z^5} - \frac{12 \theta_y y}{\Theta_0^2 z^5} - \frac{18 y^2}{z^6 \Theta_0^2} \right] \\
&\quad \times \exp[\ldots]
\end{align*}
\]
\[ \frac{\theta_0^2}{2} \frac{\partial^2 N}{\partial A_y^2} = \frac{8\sqrt{3} N_0}{\pi \theta_0^2} \left[ -\frac{1}{z^3} + \frac{4A_y^2}{\theta_0^2 z^3} - \frac{12yA_y}{\theta_0^2 z^5} - \frac{9y^2}{\theta_0^2 z^6} \right] \exp[\ldots] \]
(iii) \[ N(z, \theta_y) = \frac{4\sqrt{3} N_0}{\pi \theta_0^2 z^2} \exp \left[ \frac{-2 \theta_y^2}{\theta_0^2 z^2} \right] \int_{-\infty}^{\infty} \exp \left[ \frac{6y \theta_y}{\theta_0^2 z^2} - \frac{6y^2}{2\theta_0^2 z^2} \right] dy \]

**CALL THIS A**

Use \[ \int_{-\infty}^{\infty} e^{-a(x^2 + bx)} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{b^2/4a} \]

In our case, \( a = \frac{6}{\theta_0^2 z^2} \), \( b = -\frac{6 \theta_y}{\theta_0^2 z^2} \), so

\[ N(z, \theta_y) = A \frac{1}{2} \sqrt{\frac{\pi \theta_0^4 z^3}{6}} \exp \left( \frac{36 \theta_y^2}{\theta_0^2 z^4} - \frac{1}{4} \frac{\theta_0^2 z^3}{6} \right) \]

\[ N(z, \theta_y) = \frac{4\sqrt{3} N_0}{\pi \theta_0^2 z^2} \frac{1}{2} \frac{\sqrt{\pi \theta_0^4 z^3}}{\sqrt{3} \sqrt{2}} \exp \left[ -\frac{2 \theta_y^2}{\theta_0^2 z^2} + \frac{3 \theta_y^2}{2 \theta_0^2 z^2} \right] \]

\[ N(z, \theta_y) \, d\theta_y = \frac{N_0}{\sqrt{2\pi} \theta_0 \sqrt{z}} \exp \left[ -\frac{\theta_y^2}{2\theta_0^2 z} \right] \]

Gaussian, normalized to 1, mean \( \mu \), rms \( \sigma \) is

\[ G(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

Identify \( \mu = 0 \), \( \sigma = \theta_0 \sqrt{z} \)

and the normalization is correct, i.e. total number of particles is \( N_0 \)
(iv) Dropping the normalization now

\[ N(z, y) \, dy \sim \exp \left( -\frac{6y^2}{\Theta_0^2 z^3} \right) \int_{-\infty}^{\infty} \exp \left[ -\frac{2}{\Theta_0^2} \left( \frac{\Theta_2 y}{z} - \frac{3y \Theta_4}{z^2} \right) \right] \, dy \, dy \]

Again, use

\[ \int_{-\infty}^{\infty} e^{-(ax^2 + bx)} \, dx = \frac{1}{\sqrt{a}} \left( \frac{\pi}{2} \right) e^{\frac{b^2}{4a}} \]

In this case \( a = \frac{2}{\Theta_0^2 z} \), \( b = -\frac{6y}{\Theta_0^2 z^2} \), so

\[ N(z, y) \, dy \sim \exp \left( -\frac{6y^2}{\Theta_0^2 z^3} \right) \cdot \exp \left( \frac{26y^2}{\Theta_0^2 z^4} \right) \cdot \exp \left( \frac{26y^2}{\Theta_0^2 z^4} \right) \]

\[ N(z, y) \, dy \sim \exp \left( -\frac{3y^2}{2\Theta_0^2 z^3} \right) \]

Gaussian \( \sim \exp \left( \frac{(y - \mu)^2}{2\sigma^2} \right) \Rightarrow \mu = 0 \]

\[ \sigma^2 = \frac{\Theta_0^2 z^3}{3}, \quad \sigma = \frac{\Theta_0 z}{\sqrt{3}} \]

(v) At \( z = t \), \( y = d \)

\[ N(\theta_1) \, d\theta_1 \sim \exp \left( -\frac{2}{\Theta_0^2} \left( \frac{\Theta_2 \theta_1}{t} - \frac{3d \Theta_4}{t^2} \right) \right) \, d\theta_1 \, d\theta_1 \]

\[ \sim \exp \left[ -\frac{(2\Theta_2^2 - 6d \theta_1)}{(t \Theta_0^2)} \right] \, d\theta_1 \, d\theta_1 \]

\[ \sim \exp \left[ -\frac{(2\Theta_2^2 - 6d \theta_1 + \frac{9d^2}{2t^3 \Theta_0^2} - \frac{9d^2}{2t^3 \Theta_0^2})}{(t \Theta_0^2)} \right] \, d\theta_1 \]

(we are completing the square in the exponent)
\[ N(\theta) \, d\theta \sim \exp \left[ - \left( \frac{\sqrt{2} \theta}{\theta_0 \sqrt{t}} - \frac{3d}{\sqrt{2} t^{3/2} \theta_0} \right)^2 \right] \]

\[ N(\theta) \, d\theta \sim \exp \left[ - \left( \frac{2\theta \sqrt{t} - 3d}{\sqrt{2} t^{3/2} \theta_0} \right)^2 \right] \]

\[ N(\theta) \, d\theta \sim \exp - \frac{4t^2 (\theta - \frac{3}{2} \frac{d}{\sqrt{t}})^2}{2t^3 \theta^2} \]

\[ N(\theta) \, d\theta \sim \exp - \frac{2 (\theta - \frac{3}{2} \frac{d}{\sqrt{t}})}{t \theta^2} \]

This is gaussian with \[ \mu = \frac{3}{2} \frac{d}{\sqrt{t}} \]

\[ 2\sigma^2 = \frac{t \theta_0^2}{2} \]

\[ \sigma = \frac{\theta_0 \sqrt{t}}{2} \]
(a) Bethe-Bloch, from PDG

\[
\frac{dE}{dx} = - K \frac{z^2 Z}{A} \frac{1}{\beta^2} \left[ \frac{1}{2} \log \frac{2m_c^2 \beta^2 \gamma^2 T_{max}}{I^2} - \frac{\beta^2 - \frac{5}{2}}{2} \right]
\]

\[\gamma^2 = \frac{1}{1 - \beta^2} = \frac{\gamma^2}{\beta^2} = \gamma^2 + 1 \quad \rightarrow \quad \gamma = \sqrt{\gamma^2 + 1}
\]

\[
T_{max} = \frac{2m_c^2 \beta^2 \gamma^2}{1 + 2\gamma (m_e/m_e) + (m_e^2)}
\]

M = mass of projectile

Express everything in terms of momentum, or \( P/M \)

\[\gamma \beta = \frac{P}{M} = \gamma \]

\[
\frac{\beta^2}{1 - \beta^2} = \gamma^2 \quad \rightarrow \quad \beta^2 = \frac{\gamma^2}{\gamma^2 + 1}
\]

We are concerned with \( P, \mu, K, \rho \).

For these particles \( M \gg m_e \Rightarrow T_{max} \sim 2m_c^2 \beta^2 \gamma^2 \)

So

\[
\frac{dE}{dx} = - \frac{K \gamma^2}{A} \frac{\gamma^2 + 1}{\gamma^2} \left[ \frac{1}{2} \log \frac{4m_e^2 \gamma^4}{I^2} - \frac{\gamma^2}{\gamma^2 + 1} \right]
\]

Using \( I = 16 \text{eV} \), plot \( \frac{dE}{dx} \) in arbitrary units
(b) Ignoring the bend
\[ t = \frac{R}{\beta c} \]
Again let \( \eta = \frac{P}{M} \), then \( \beta = \frac{C}{\sqrt{\eta^2 + 1}} \)

\[ t = \frac{\sqrt{\eta^2 + 1}}{\eta} \frac{R}{c} \quad \text{with} \quad R = 1.5 \text{m} \quad \text{and} \quad c = 3 \times 10^8 \text{m/s} \]

Including the bend, the path-length is not \( R \).
The radius of curvature \( \rho \) is given by \( \rho = \frac{P}{0.3B} \)
where \( P \) is in GeV/c, \( \rho \) is in m, \( B \) is in Tesla.

\[ L = \rho \alpha \]
\[ R^2 = \rho^2 + \rho^2 - 2 \rho^2 \cos \alpha \]
\[ R^2 = 2 \rho^2 (1 - \cos \alpha) \]
\[ R^2 = 2 \rho^2 \frac{\alpha^2}{2} \]
\[ \sin \frac{\alpha}{2} = \frac{R}{2\rho} \]
\[ \alpha = 2 \sin^{-1} \frac{R}{2\rho} \]

So \( L = 2\rho \sin^{-1} \frac{R}{2\rho} \), and \( t = \frac{\sqrt{\eta^2 + 1}}{\eta} \frac{L}{c} \)

\[ t = 2 \frac{\sqrt{\eta^2 + 1}}{\eta} \frac{\rho}{c} \sin^{-1} \frac{R}{2\rho} \quad \text{with} \quad \rho = \frac{P}{0.3B} \]
NO BEND

WITH BEND

\[ \text{TOF (nsec)} \]

\[ P \text{ (GeV/c)} \]
(a) Probability \( P(x) \, dx = \frac{dx}{D} \), \(-\frac{D}{2} < x < \frac{D}{2}\)

\[ P(x) = 0 \quad x > \frac{1}{2} \]

Note this is properly normalized \( \int_{-\infty}^{\infty} P(x) \, dx = 1 \)

\( [P(x) \, dx \text{ here is the probability that the charged particle goes through the detector between } x \text{ and } x+dx \text{ given that the channel hit was centered at } x=0] \)

\[ \sigma^2 = \int_{-\infty}^{\infty} P(x) (x-<x>)^2 \, dx \quad \text{but } <x> = 0 \]

\[ \sigma^2 = \int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{x^2}{D} \, dx = \frac{1}{3D} \left( \left( \frac{D}{2} \right)^3 + \left( \frac{D}{2} \right)^3 \right) = \frac{D^2}{12} \]

\[ \Rightarrow \sigma = \frac{D}{\sqrt{12}} \]

(b)

\[ Z(R) = A + BR \]

\[ \begin{cases} \begin{align*} Z_4 &= A + BR_4 \\ Z_2 &= A + BR_2 \end{align*} \end{cases} \]

\[ \begin{cases} \begin{align*} B &= (Z_2 - Z_1) / (R_2 - R_1) \\ A &= Z_2 - (Z_2 - Z_1) \frac{R_2}{R_2 - R_1} = \frac{Z_2 R_2 - Z_2 R_1 - Z_3 R_0 + Z_4 R_2}{R_2 - R_1} \end{align*} \end{cases} \]

\[ A = \frac{R_2}{R_2 - R_1} Z_1 - \frac{R_1}{R_2 - R_1} Z_2 \]
\[ Z(R=0) = A = \frac{R_2}{R_2 - R_1} z_1 - \frac{R_1}{R_2 - R_1} z_2 \]

Let \( \sigma^2 = \text{Resolution in } Z(R=0) \)

\[ \sigma^2 = \left( \frac{R_2}{R_2 - R_1} \right)^2 \sigma^2(z_1) + \left( \frac{R_1}{R_2 - R_1} \right)^2 \sigma^2(z_2) \]

But \( \sigma^2(z_1) = \sigma^2(z_2) = \frac{D^2}{12} \)

Therefore

\[ \sigma^2 = \frac{D^2}{12} \frac{R_1^2 + R_2^2}{(R_2 - R_1)^2} \]

\[ \sigma = \frac{D}{\sqrt{12}} \frac{\sqrt{R_1^2 + R_2^2}}{R_2 - R_1} \]

Or, writing \( r = R_2/R_1 \),

\[ \sigma = \frac{D}{\sqrt{12}} \frac{\sqrt{1 + r^2}}{r - 1} \]
Let $z=0$ origin point of particle
$z_f$ coordinate at beam pipe
$z_\ell$ coordinate at first layer
$z_\ell' = \text{coordinate at 2nd layer}$
$\theta_0 = \text{scattering angle at beam pipe}$
$\theta_1 = \text{scattering angle at 1st layer}$

$Z_0 = \frac{R_0}{\tan \alpha}$
$Z_\ell = Z_0 + (R_1 - R_0) \tan(\alpha - \theta_0) \approx Z_0 + (R_1 - R_0) (\tan \alpha - \theta_0)$ small $\theta_0$
$Z_\ell \approx \frac{R_0}{\tan \alpha} + R_1 \tan \alpha - R_0 \tan \alpha - (R_1 - R_0) \theta_0$
$Z_\ell \approx R_1 \tan \alpha - (R_1 - R_0) \theta_0$ \hspace{1 cm} (1)

Similarly
$Z_2 \approx R_2 \tan \alpha - (R_2 - R_0) \theta_0 - (R_2 - R_1) \theta_1$ \hspace{1 cm} (2)

From part (b)
$Z(0) = \frac{R_2}{R_2 - R_1} Z_1 - \frac{R_1}{R_2 - R_1} Z_2$ \hspace{1 cm} (3)

$\sigma^2(Z_\ell) = \frac{D}{12} + (R_1 - R_0) \sigma_\theta^2$

\text{MS contribution}
\[ \sigma^2(z_2) = \frac{D}{12} + (R_2 - R_0)^2 \sigma_{\theta_0}^2 + (R_2 - R_1)^2 \sigma_{\theta_1}^2 \]

Note that in order to get the uncertainty in \( z(0) \)
I cannot do as in part (b) and write

\[ \sigma^2(z=0) = \left( \frac{R_2}{R_2 - R_1} \right)^2 \sigma^2(z_1) + \left( \frac{R_1}{R_2 - R_1} \right)^2 \sigma^2(z_2) \]

NO!

This is because now \( z_1 \) and \( z_2 \) are correlated.
What I can do is write down

\[ \sigma^2(z=0) = \sigma^2(\text{measurement}) + \sigma^2(\text{MS}) \quad (4) \]

where \( \sigma^2(\text{measurement}) = \frac{D^2}{12} \frac{1 + r^2}{(1-r)^2} \) as in (b).

Then to calculate \( \sigma^2(\text{MS}) \), let me write down the deviation from \( z=0 \) due to multiple scattering.
Deviation (from equations (1), (2), (3))

\[ \delta z_0 = \frac{R_2}{R_2 - R_1} \left[ - (R_2 - R_0) \theta_0 \right] - \frac{R_1}{R_2 - R_1} \left[ - (R_2 - R_0) \theta_0 - (R_2 - R_1) \theta_1 \right] \]

\[ \delta z_0 = -R_1 R_2 \theta_0 + R_0 R_2 \theta_2 + R_1 R_2 \theta_1 - R_1 R_0 \theta_0 + R_1 R_2 \theta_1 - R_2^2 \theta_1 \]

\[ \delta z_0 = \frac{R_0 (R_2 - R_1)}{R_2 - R_1} \theta_0 + \frac{R_1 (R_2 - R_1)}{R_2 - R_1} \theta_1 = R_0 \theta_0 + R_1 \theta_1 \]

\[ \sigma_{\text{MS}}^2 = R_0^2 \sigma_{\theta_0}^2 + R_1^2 \sigma_{\theta_1}^2 \]

Therefore, equation (4)

\[ \sigma^2 = \frac{D^2}{12} \frac{1 + r^2}{(1-r)^2} + R_0^2 \sigma_{\theta_0}^2 + R_1^2 \sigma_{\theta_1}^2 \]
(d) $R_0 = 2 \text{ cm} \quad R_1 = 3 \text{ cm} \quad R_2 = 4 \text{ cm} \quad t = \frac{4}{3} \quad D = 50 \mu m$

$\bar{\theta} = \frac{13.6 \text{ MeV}}{\beta P} \sqrt{N}$ where $N =$ number of radiation lengths

This equation is from page 198 of the 2002 PDG, neglecting the small log term.

Beau pipe is 1.2 mm thickness Be.

Be is good because it is low Z, so long X

Silicon is 300 $\mu m$ thick

$\begin{align*}
(X(\text{Be})) &= 35.28 \text{ cm} \quad \text{PDG, page 84} \\
(X(s.i)) &= 9.36 \text{ cm} \quad \text{PDG, page 84}
\end{align*}$

$\begin{align*}
(N(\text{Be})) &= 1.2 \text{ mm} / 35.28 \text{ cm} = 3.4 \times 10^{-3} \\
(N(s.i)) &= 0.3 \text{ mm} / 93.6 \text{ mm} = 3.2 \times 10^{-3}
\end{align*}$

With no multiple scattering:

$\sigma = \frac{D}{\sqrt{12}} \sqrt{\frac{1+16/9}{16q-1}} = \frac{D}{\sqrt{12}} \frac{5/3}{7/9} = \frac{15D}{7\sqrt{12}} = \frac{15}{7} \frac{50}{\sqrt{12}} \mu m$

$\sigma = 31 \mu m$

With multiple scattering:

$\sigma^2 = (31 \mu m)^2 + R_0^2 \sigma_0^2 + R_1^2 \sigma_1^2$

Plugging the numbers in:

$\sigma^2 = (31 \mu m)^2 + 4 \text{ cm}^2 \left(\frac{13.6 \text{ MeV}}{\beta P}\right)^2 3.4 \times 10^{-3} + 9 \text{ cm}^2 \left(\frac{13.6 \text{ MeV}}{\beta P}\right)^2 2.2 \times 10^{-3}$

$\sigma^2 = (31 \mu m)^2 + \left(\frac{28 \text{ cm} \text{ GeV}}{\beta P}\right)^2$
\[ \frac{dE}{dx}_{\text{min}} \sim 1.45 \text{ MeV cm}^2/\text{g} \]

\[ P = \frac{7.87 \text{ g}}{\text{cm}^2} \Rightarrow \frac{dE}{dx}_{\text{min}} = 11.4 \text{ MeV/cm} \]

Kinetic Energy \( T = E - m = \sqrt{P^2 m^2} - m \)

In order to lose all the kinetic energy we must have a thickness

\[ \Delta x \sim T \frac{dE}{dx}_{\text{min}} = (\sqrt{P^2 m^2} - m) \frac{dE}{dx}_{\text{min}} \]

\[ P = 1 \text{ GeV} \Rightarrow \Delta x = \frac{\sqrt{1 + (0.106)^2} - 0.106}{11.4 \times 10^{-3}} \text{ cm} \]

\[ \Delta x \approx 80 \text{ cm} \]

\[ P = 5 \text{ GeV} \Rightarrow \Delta x = \frac{5 - 0.106}{11.4 \times 10^{-3}} \text{ cm} \]

\[ \Delta x = 430 \text{ cm} \]

\[ P = 10 \text{ GeV} \Rightarrow \Delta x = \frac{10 - 0.106}{11.4 \times 10^{-3}} \text{ cm} \]

\[ \Delta x = 870 \text{ cm} \]

Also \( \lambda_I \approx 132 \text{ g/cm}^2 \Rightarrow \lambda_I = \frac{132}{7.84} \text{ cm} \approx 17 \text{ cm} \)

For \( \Delta x = 80 \text{ cm} \)

\[ e^{-\frac{80}{17}} \sim 0.1 \% \]

\[ e^{-430/17} \sim 10^{-11} \rightarrow \text{ about right} \]

\[ e^{-870/17} \sim 10^{-22} \rightarrow \text{ unreasonable small other effects come into play} \]
Let $\lambda = \frac{P}{M} \gamma = \beta c \gamma = \text{mean distance travelled by } \pi^0 \text{ before decay.}$

For a $\pi^0$ produced at $x$, the probability that it decays at $y$ is

$$p(y) \, dy = e^{-\frac{y}{\lambda}} \, dy$$

Then suppose $\pi^0 \rightarrow \pi \pi$ (this is almost unit probability)

The probability that at least one of the $\pi$'s converts is

$$2 \left( 1 - e^{-\frac{x-y-x}{\lambda}} \right) \quad \text{for } x_0 > x$$

Then if $\pi^0$ are produced at a constant rate $k \, dx$ uniformly in $x$, the rate of $e^+e^-$ pairs from $\pi^0 \rightarrow \pi \pi \rightarrow e^+e^-$ is

$$R(t) = \int \int k \, dx \, e^{-\frac{y}{\lambda}} \, dy \, \frac{z(t-y-x)}{x_0}$$

where I neglected the difference between 99.8% and 100%
\[ R(t) = \int_0^t dx \frac{2k}{\lambda X_0} \int_0^{t-x} dy \, (t-y-x) \, e^{-y/\lambda} \]

\[ R(t) = \frac{2k}{\lambda X_0} \int_0^t dx \left\{ (t-x) \int_0^{t-x} e^{-y/\lambda} \, dy - \int_0^{t-x} y \, dy \, e^{-y/\lambda} \right\} \]

\[ R(t) = \frac{2k}{\lambda X_0} \int_0^t dx \left\{ (t-x) \left( \lambda - \lambda e^{-(a-t)/\lambda} \right) - \int_0^{t-x} dy \, ye^{-y/\lambda} \right\} \]

For the second integral, use \( \int dx \, xe^{ax} = \frac{e^{ax}}{a} (x - \frac{1}{a}) \)

So \( \int_0^{t-x} ye^{-y/\lambda} = -\lambda \left[ e^{-y/\lambda} (y + \lambda) \right]_{0}^{t-x} \)

\[ = -\lambda \left[ e^{-(a-t)/\lambda} (t-x+\lambda) - \lambda \right] \]

\[ R(t) = \frac{2k}{\lambda X_0} \int_0^t dx \left\{ \lambda(t-x) - \frac{(a-t)/\lambda}{e^{(a-t)/\lambda}} - \left( t-x+\lambda \right) \right\} \]

\[ R(t) = \frac{2k}{\lambda X_0} \int_0^t dx \left\{ \frac{\lambda t - \lambda}{e^{(a-t)/\lambda}} + \lambda \right\} \]

\[ R(t) = \frac{2k}{X_0} \int_0^t dx \left( t-x-\lambda \right) + \frac{2k}{X_0} \int_0^t dx \lambda \left( e^{(a-t)/\lambda} - \lambda \right) \]

\[ R(t) = \frac{2k}{X_0} \left[ t^2 - \frac{1}{2} \frac{t^2}{\lambda} - \lambda t \right] + \frac{2k}{X_0} \lambda e^{-t/\lambda} \int_0^t dx \, e^{x/\lambda} \]

\[ R(t) = \frac{k}{X_0} \left[ t^2 - 2\lambda t \right] + \frac{2k\lambda^2}{X_0} e^{-t/\lambda} \left( e^{t/\lambda} - 1 \right) \]
\[ R(t) = \frac{2kt}{X_0} \left[ \frac{t}{2} - \lambda + \frac{\lambda^2}{t} \left(e^{\frac{t}{\lambda}} - 1\right)e^{-\frac{t}{\lambda}} \right] \]

This neglects the contribution of \( \pi^0 \rightarrow e^+e^-\gamma \) decays.

They contribute at a rate \( B \cdot k \).

So

\[ R(t) = 2kt \left\{ \frac{B}{2} + \frac{d}{X_0} \left[ \frac{t}{2} - \lambda + \frac{\lambda^2}{t} \left(1 - e^{-\frac{t}{\lambda}}\right) \right] \right\} \]