Scattering in Toy Theory

$AA \rightarrow BB$

The fundamental vertex is

1st rule: labels

2nd rule: coupling constant

3rd rule: propagator

4th rule: $\delta$-functions

5th rule: integration of internal momenta

6th rule: get rid of last $\delta$-function \( \leftarrow \) we'll do this at the end

\[
-i M' = (-i\hbar)^2 \frac{(2\pi)^4}{(2\pi)^4} \delta^4(p_1 - p_3) \delta^4(p_2 + q - p_4) \int \frac{d^4q}{(q^2 - m_c^2)}
\]

Use the first $\delta$-function to do the integral:

\[
-q = p_1 - p_3
\]

\[
-i M' = -i \frac{q^2}{(p_1 - p_3)^2 - m_c^2} \frac{(2\pi)^4}{(2\pi)^4} \delta^4(p_2 + (p_1 - p_3) - p_4)
\]
The $\delta$-function which remains is

$$(2\pi)^4 \delta^4 \left( \frac{P_1 + P_2 - P_3 - P_4}{P_1 - P_3} \right)$$

INITIAL FINAL

This is the overall conserving $\delta$-function which rule 6 says we should get rid of. So we are left with

$$-iM = -\frac{iq^2}{(P_1 - P_3)^2 - M^2 c^2}$$

Note

$$q^2 = (P_3 - P_1)^2 = (P_4 - P_2)^2 = (P_2 - P_4)^2$$

\[ P_1 = P_3 + q \]
\[ P_1 + q = P_3 \]
\[ q + P_2 = P_4 \]
\[ P_2 = q + P_4 \]

In other words, you can put the arrow on $q$ \(\rightarrow\) in either of the two directions, it doesn't matter.

Also, you can take $q$ as made up of $P_i$'s at any of the two vertices, it doesn't matter.

But is this the whole story?

NO Because I could have interchanged the labels on the particles in the final state.
We had

\[ P_1 \rightarrow A, \quad B \rightarrow P_2, \quad A \rightarrow B, \quad P_4 \rightarrow P_3. \]

We can also have

\[ P_1 \rightarrow A, \quad B \rightarrow P_2, \quad A \rightarrow B, \quad P_4 \rightarrow P_3. \]

Sometimes (I don't much like it) written as

There will be an \( M_i \) associated with each one of these.

The first one we already calculated

\[ M = \frac{g^2}{(P_i - P_3)^2 - M_c^2 c^2} \]

The second one is easy to calculate, just interchange \( P_3 \leftrightarrow P_4 \).

So

\[ M = \frac{g^2}{(P_i - P_3)^2 - M_c^2 c^2} + \frac{g^2}{(P_i - P_4)^2 - M_c^2 c^2} \]
Note one important thing:

When there are no internal loops, you can just skip rules 4, 5, 6. This is because the integration over internal momenta is then trivial.

\[ \text{Set } c = 1 \]

\[ P_1 = \begin{pmatrix} E \cr p \cr 0 \cr 0 \end{pmatrix} \]
\[ P_2 = \begin{pmatrix} E \cr -p \cr 0 \cr 0 \end{pmatrix} \]
\[ P_3 = \begin{pmatrix} E' \cr p' \cos \theta \cr p' \sin \theta \cr 0 \end{pmatrix} \]
\[ P_4 = \begin{pmatrix} E' \cr -p' \cos \theta \cr -p' \sin \theta \cr 0 \end{pmatrix} \]

\[ E = \sqrt{p^2 + m_A^2} \quad E' = \sqrt{p'^2 + m_B^2} \]

\[ (P_1 - P_3)^2 = p^2 + p_3^2 - 2p_1p_3 = m_A^2 + m_B^2 - 2EE' + 2pp' \cos \theta \]
\[ (P_1 - P_4)^2 = p^2 + p_4^2 - 2p_1p_4 = m_A^2 + m_B^2 - 2EE' - 2pp' \cos \theta \]

**Special case** \( m_A = m_B = m \quad m_C = 0 \quad \Rightarrow \quad p = p' \quad E = E' \)

\[ (P_1 - P_3)^2 = 2m^2 + 2E^2 - 2p^2 \cos \theta = 2m^2 - 2(p^2 + m^2) - 2p^2 \cos \theta \]
\[ (P_1 - P_3)^2 = -2p^2 (1 - \cos \theta) \]

Similarly \( (P_1 - P_4)^2 = -2p^2 (1 + \cos \theta) \)

Then

\[ M = \frac{g^2}{-2p^2 (1 + \cos \theta)} + \frac{g^2}{-2p^2 (1 - \cos \theta)} = -\frac{g^2}{2p^2} \frac{1 - \cos \theta + 1 + \cos \theta}{1 - \cos^2 \theta} \]

\[ M = -\frac{g^2}{p^2 \sin^2 \theta} \quad \left( M \right)^2 = \frac{g^4}{p^4 \sin^4 \theta} \]
Then \[
\frac{d\sigma}{d\Omega} = \frac{\hbar^2 c^2}{64 \pi^2} \frac{\mathcal{S} \left\{ \frac{1}{p_1^4} \right\} \left\{ \frac{1}{p_2^4} \right\}}{(E_1+E_2)^2} = 4E^2
\]

\[
\frac{d\sigma}{d\Omega} = \frac{\hbar^2 c^2 g^4}{512 \pi^2 E^2 p^4 \sin^4 \alpha}
\]
Note the factor \( g^4 = g^{2N} \)
where \( N = \) number of vertices

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On the road towards Quantum Electrodynamics

1st step: Dirac equation

Dirac equation is a generalization of Schrödinger equation to a relativistic situation, valid for \( S=\frac{1}{2} \) particles.
Go back to where Schrödinger equation came from.
Free particle (no potential energy)
\[
E = K.E.
E = \frac{p^2}{2m}
\]
QM \[
E \rightarrow i\hbar \frac{d}{dt}, \quad p \rightarrow -i\hbar \nabla
\]
\[
\Rightarrow -\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{d\psi}{dt}
\]
↑ note how time and space are on equal footing (1st derivative vs 2nd derivative)
But now try relativistically
\[ E = m^2 c^4 + p^2 c^2 \]
\[ p^\mu p_\mu = m^2 c^2 \quad \Rightarrow \quad \frac{p}{\sqrt{E^2 - p^2 c^2}} \]

\[ p_\mu \rightarrow i \hbar \frac{d}{dx_\mu} \quad \Rightarrow \quad \frac{d}{dx} = \left( \frac{1}{c} \frac{\partial}{\partial t} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) \]

\[ \Rightarrow \quad p_\mu p^\mu = -\hbar^2 \frac{d^2}{dt^2} = -\frac{\hbar^2}{c^2} \frac{d^2}{dt^2} + \frac{\hbar^2 \delta^2}{dx^2} + \frac{\hbar^2 \delta^2}{dy^2} + \frac{\hbar^2 \delta^2}{dz^2} \]

\[ \hbar^2 \nabla^2 \]

so \[ p_\mu p^\mu = m^2 c^2 \]

becomes

\[ \left[ -\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = \frac{m^2 c^2 \psi}{\hbar^2} \right] \quad \text{Klein Gordon equation} \]

Note how \( t \) and \( x \) are on equal footing here.

When this was first proposed a couple of problems came up immediately

1. This has negative \( E \) solution

   Basically because \( E^2 = m^2 c^4 + p^2 c^2 \)

   \[ E = \pm \sqrt{m^2 c^4 + p^2 c^2} \]

2. Also has negative probabilities

   Looked like a disaster. Now we know how to handle these. It is not a real problem, I don't want to
really go into this. The KG equation is actually a
fine relativistic wave equation for describing spinless particles.
Dirac invented a new equation to "solve" the problem
with KG. It so happens that this equation is just what
you need to describe $S = \frac{1}{2}$ particles (and antiparticles too).
Since $\nu$ leptons, quarks have $S = \frac{1}{2}$, the Dirac egtn turns
out to be more important.
The problem with KG was that it had second order
derivatives. Dirac tried to write down an egtn with 1st
order derivatives only.

\[
\mathbf{E}^2 c^4 \quad H \psi = \left( \bar{\psi} \left( \bar{\gamma} \cdot \mathbf{P} + \gamma \cdot \beta mc \right) \psi \right)
\]

But then we require $H^2 \psi = (\mathbf{P} + m^2 c^2) \psi$

(\[\mathbf{P} \quad \text{and} \quad M \quad \text{are operators here} \]

\[\mathbf{P} = -i \hbar \nabla \]
\[\alpha \quad \text{and} \quad \beta \quad \text{are some parameters that we need}
\]
\[\text{to determine} \]

\[H^2 \psi = \left( \sum_i \alpha^2 P_i^2 + \sum_{i \neq k} (\alpha_i \beta \gamma + \alpha_k \beta \gamma) P_i P_k + \sum_i (\alpha_i \beta + \beta \gamma) P_i mc \right. \]
\[\left. + \beta^2 m^2 c^2 \right) \psi \]
We want this to be \( (P^2 + m^2c^2)^4 = \left( \sum_i P_i^2 + m^2c^2 \right)^4 \).

Immediately we see that we want \( \alpha_i^2 = 1 \)  \( \beta^2 = 1 \)

But there is a problem with the cross terms. How can we make them be zero?

Dirac had a stroke of genius.

Let's take the \( \alpha \) and \( \beta \) to be not numbers but matrices.

Then I can have, say \( \alpha_1^2 = 1 \), \( \alpha_2^2 = 1 \), but \( \alpha_1 \alpha_2 + \alpha_2 \alpha_1 = 0 \).

So the requirement becomes
\[
\alpha_i^2 = 1 \\
\beta^2 = 1 \\
\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad \text{if} \quad i \neq j \\
\alpha_i \beta + \beta \alpha_i = 0.
\]

Next question: What is the dimensionality of these matrices.

Dirac showed that it has to be at least \( 4 \times 4 \) - And that is what he picked.

The choice of these matrices is not unique.

The physics does not change if we choose a different set but the algebra does. A common choice is the Pauli-Dirac representation
\[
\alpha' = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

(each one of these is a 2x2 matrix)

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
Then the Dirac equation is

\[ H \psi = i \hbar \frac{\partial \psi}{\partial t} \]

\[ H = -i \hbar c \vec{\alpha} \cdot \vec{p} + \beta m c^2 \]

\[ -i \hbar c \vec{\alpha} \cdot \vec{\nabla} \psi + \beta m c^2 = i \hbar \frac{\partial \psi}{\partial t} \]

Multiply both sides by \( \beta \), rearrange terms

\[ + i \hbar \beta \vec{\alpha} \cdot \vec{p} \psi + i \beta \hbar \frac{\partial \psi}{\partial t} - mc \psi = 0 \]

In more compact notation, introduce set of \( \gamma^\mu \)

\[ \gamma^\mu = (\beta, \beta \vec{\alpha}) \]

i.e. \( \gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) in Pauli-Dirac representation

\[ \gamma^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ i \hbar \gamma^\mu \frac{\partial}{\partial x^\mu} \psi - mc \psi = 0 \]

But what does this mean? \( \gamma^\mu \) are 4x4 matrices

\( \psi \) is a 4-element column vector

\[ \psi = \begin{pmatrix} \psi_1(x^1, t) \\ \psi_2(x^2, t) \\ \psi_3(x^3, t) \\ \psi_4(x^4, t) \end{pmatrix} \]
Solutions of Dirac Equation

First for simplicity look at particle at rest
\[ \mathbf{P} = 0 \Rightarrow \nabla \psi = 0, \quad \psi \text{ independent of position} \]

\[ i \hbar \gamma \frac{d}{dt} \psi - m c \psi = 0 \]

becomes

\[ \frac{i \hbar}{c} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_A}{\partial t} \\ \frac{\partial \psi_B}{\partial t} \end{pmatrix} - m c \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0 \]

\[ \frac{i \hbar}{c} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_A}{\partial t} \\ \frac{\partial \psi_B}{\partial t} \end{pmatrix} = 0 \]

\[ - \frac{i \hbar}{c} \frac{\partial \psi_B}{\partial t} + m c \psi_B = 0 \]

\[ \frac{\partial \psi_A}{\partial t} = -i \left( \frac{m c^2}{\hbar} \right) \psi_A \]

\[ \frac{\partial \psi_B}{\partial t} = i \left( \frac{m c^2}{\hbar} \right) \psi_B \]

\[ \psi_A(t) = C e^{-i \left( \frac{m c^2}{\hbar} \right) t} \]

\[ \psi_B(t) = C e^{i \left( \frac{m c^2}{\hbar} \right) t} \]

So, since \( m c^2 = E > 0 \)

\[ \psi_A = C e^{-\frac{i c}{\hbar} E t} \]

\[ \psi_B = C e^{-\frac{i c}{\hbar} (-E) t} \]

\[ \text{positive energy solution} \]

\[ \text{negative energy solution} \]