Group Theory in a Nutshell

Set $G \{a, b, c, \ldots\}$ is a group if

1. Operation $\cdot$ such that
   - Group multiplication

   1. $a, b \in G$, $a \cdot b \in G$ (closure)
   2. $(ab)c = a(bc)$ (associative)
   3. $\exists e \in G$, $\forall a \in G$, $ae = ea = a$ (identity exists)
   4. $\forall a \in G$, $\exists a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse exists)

Example: $G = \mathbb{R}$, group multiplication = multiplication of numbers excluding 0, because it has no inverse

Lie Group

$\infty$ number of group members

$\vec{\Theta} = (\Theta_1, \Theta_2, \ldots, \Theta_n)$

$\uparrow$ real parameters

$\exists \infty$ number of $\vec{\Theta}$

$\vec{\Theta}$ are used to parameterize (define?) group element, i.e.

$a \in G$, $a = a(\vec{\Theta}) = a(\Theta_1, \ldots, \Theta_n)$

i.e. if you specify $\vec{\Theta}$, you specify $a$

$\vec{\Theta}$ must be such that $a(\vec{0}) = e$ and $a(\vec{\Theta}) \cdot a(\vec{\Phi}) = a(\vec{\Theta} + \vec{\Phi})$

$\vec{\Theta}(\vec{\Theta}', \vec{\Phi})$ continuously differentiable

Formal definition for our purposes, any sane $\vec{f}$ will do!
Group Representation

Mapping $G \rightarrow$ operators in linear vector space $V$
Usually represented as $N \times N$ matrices, where $N =$
dimension of $V$

I.e. $a \in G$ is mapped into some matrix $(\cdots)$

Eg $e$ is mapped into $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Group multiplication $\rightarrow$ Matrix Multiplication

We care about groups where elements are sym. transformations

Vector space that we are interested in

= Vector space of particle states

We will care more about the representations than the groups themselves

Irreducible vs Reducible Representation

Irreducible: no orthogonal subspaces of $V$ which are
invariant under ALL group operations

Eg suppose we have a 5-D representation and
all operators are of the form

$\begin{pmatrix}
  \times & \times & \times & \circ & \circ \\
  \times & \times & \times & \circ & \circ \\
  \times & \times & \times & \circ & \circ \\
  \circ & \circ & \circ & \times & \times \\
  \circ & \circ & \circ & \times & \times 
\end{pmatrix}$
Then the subspaces
(i) with basis vectors = the first three basis vectors
(ii) with basis vectors = the last two basis vectors
form invariant subspaces
The 5D representation is not irreducible, (so it is reducible)
The 5D representation can be reduced into
two representations of dimensions 3 and 2.
These representations will be irreducible and have some properties.
We write this as $5 = 3 \oplus 2$

**[Rotation Group]**: **ANGULAR MOMENTUM** from different perspectives.

We said that $J_x, J_y, J_z$ generates rotations around
$x-$ (y-, z-) axes.
In general, a rotation in 3D can be generated by

$$U(\vec{\alpha}) = e^{-i\vec{\alpha} \cdot \vec{J}}$$

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$$

$$\vec{\alpha} \cdot \vec{J} = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3$$

$\vec{\alpha}$ encodes the details of rotation, e.g. the 3-Euler angles.

Rotational Invariance

$$[U, H] = 0$$

$$[J_i, H] = 0$$

This is an example of a Lie group.
\[
\left[ J_i, J_j \right] = i \varepsilon_{ijk} J_k = i \varepsilon_{ijk} J_k \quad \text{sum over } k \text{ implied}
\]

\( \varepsilon_{ijk} \): Levi-Civita symbol

\( \varepsilon_{ijk} = 1 \) if \( ijk = 123, 231, 312 \) (i.e. permutations of 123)

\( \varepsilon_{ijk} = -1 \) if \( ijk = 132, 213, 321 \)

\( \varepsilon_{ijk} = 0 \) otherwise

(Note \( \varepsilon_{ijk} = -\varepsilon_{jki} \) as it should, see commutator)

\[
\left[ J_i, J_j \right] = i \varepsilon_{ijk} J_k \quad \text{Lie Algebra of generators}
\]

Aside

If I have two operators that do not commute,
these operators cannot have common eigenvectors

\( A \left| 1^+ \right\rangle = a \left| 1^+ \right\rangle \)
\( B \left| 1^+ \right\rangle = b \left| 1^+ \right\rangle \)

Suppose \( [A, B] \neq 0 \)
\[
[A, B] \left| 1^+ \right\rangle = AB \left| 1^+ \right\rangle - BA \left| 1^+ \right\rangle = (ab - 6a) \left| 1^+ \right\rangle = 0
\]

But \( [A, B] \left| 1^+ \right\rangle = C \left| 1^+ \right\rangle 
eq 0 \) in general

A given representation of the rotation group cannot have basis vector that are simultaneous eigenvectors of \( J_x, J_y, J_z \) (or \( J_1, J_2, J_3 \) in equivalent notation)
We choose eigenvectors of $J_3$

$J_3 \mid \Psi \rangle = m \mid \Psi \rangle$

Also $J^2 = \Sigma J^2_i \begin{bmatrix} J^2 & J_i \end{bmatrix}$

$\Rightarrow$ eigenvectors of $J_3$ are also eigenvectors of $J_i$

Can show $J^2 \mid \Psi \rangle = J (J+1) \mid \Psi \rangle$

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$J_3 \mid \Psi \rangle = m \mid \Psi \rangle$

$m = -j, -j+1, \ldots, j-1, j$ (2j+1) possibilities

You knew that already - (I hope)

Result

- Irreducible representations of rotation group have dimension $2j+1$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

- Basis vectors labelled by an integer $m = -j, -j+1, \ldots, j-1, j$

We label the eigenvectors by $j$ and $m$, i.e. we write them as $|j \ m \rangle$

**Combining Representation $\leftrightarrow$ Adding Angular Momentum**

Consider a new vector space with basis vector:

$|j_A \ j_B \ m_A \ m_B \rangle = |j_A \ \Omega M_A \rangle \ |j_B \ m_B \rangle$
The operators $\vec{J} = \vec{J}_A + \vec{J}_B$ also satisfy Lie Algebra
($\vec{J}_A$ operates on vectors with subscript $A$, $\vec{J}_B$ operates on
vectors with subscript $B$)

Then, we can construct a new representation of the rotation
group in this new vector space.
We would say that we "combined representations" we sometimes write it as

$$(2J_A + 1) \times (2J_B + 1)$$

Dimensionality of this new vector space: $N_A \cdot N_B$

In general this new representation can be reduced.

Result

new representation reduced to irreducible representations
which will be labelled by $J, M$ with

$|J_A - J_B| \leq J \leq |J_A + J_B|$

$M = -J, -J+1, \ldots, J-1, J$

These are exactly the rules for the addition of angular
momentum. They can be derived just from the
algebra of the generators of the Lie group in a
seemingly very abstract way.

Combining representations $\rightarrow$ adding angular momenta
States of definite angular momentum \( J \rightarrow \) form an irreducible representation of the rotation group

For example

\[ J_A = \frac{1}{2}, J_B = \frac{1}{2} \]

we know adding angular momentum \( J \) with \( \frac{1}{2} \) gives \( \frac{3}{2} \)

We write it as

\[
\begin{array}{c}
\begin{array}{c}
2 \otimes 2 = 3 \oplus 1 \\
(2 \frac{1}{2} + 1) \oplus (2 \frac{1}{2} + 1)
\end{array}
\end{array}
\]

How do we build these states \( |J M\rangle \)?

Third component adds \( M = m_A + m_B \)

Clearly

\[ |1 \ 1\rangle = |\frac{1}{2} \ \frac{1}{2}\rangle _A |\frac{1}{2} \ \frac{1}{2}\rangle _B \]

\[ |1 \ -1\rangle = |\frac{1}{2} \ -\frac{1}{2}\rangle _A |\frac{1}{2} \ -\frac{1}{2}\rangle _B \]

But what about \( |1 \ 0\rangle \) and \( |0 \ 0\rangle \)?

They must be linear combination of

\[ |\frac{1}{2} \ \frac{1}{2}\rangle _A |\frac{1}{2} \ -\frac{1}{2}\rangle _B \text{ and } |\frac{1}{2} \ -\frac{1}{2}\rangle _A |\frac{1}{2} \ +\frac{1}{2}\rangle _B \]

But what linear combinations?
Build step-up and step-down operators

\[ J_\pm = (J_A)_\pm + (J_B)_\pm \]

\[
(J_A)_\pm = (J_1)_A - i (J_2)_A
\]

\[
(J_A)_\pm |d_A m_A\rangle = \sqrt{\delta_A (d_A+1) - m_A (m_A-1)} |d_A m_A - 1\rangle
\]

**Key concept**

- I know \( |1 1\rangle \) belongs to the 3D \((J=1)\) representation which is irreducible

- \( J_- |1 1\rangle \) must belong to the same representation because \( J_- \) is built up of generators of the rotation group, and the 3D subspace is invariant

\[
J_- |1 1\rangle = (J_A)_{\frac{1}{2} \frac{1}{2}} |\frac{1}{2} \frac{1}{2}\rangle_A + |\frac{1}{2} \frac{1}{2}\rangle_B\]

\[
J_- |1 1\rangle = \frac{1}{\sqrt{2}} |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B + \frac{1}{\sqrt{2}} |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B
\]

This combination must belong to the \( J=1 \) representation. It has \( M=0 \) \( \Rightarrow \) must be \( |1 0\rangle \)

\[
|1 0\rangle = \frac{1}{\sqrt{2}} |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B + \frac{1}{\sqrt{2}} |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B
\]

The orthogonal combination must be \( |0 0\rangle \)

\[
|0 0\rangle = \frac{1}{\sqrt{2}} |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B - \frac{1}{\sqrt{2}} |\frac{1}{2} \frac{1}{2}\rangle_A |\frac{1}{2} \frac{1}{2}\rangle_B
\]
In general

\[ |J M\rangle = \sum_{m_A m_B} C(m_A, m_B, J, M) |J_A m_A\rangle |J_B m_B\rangle \]

Clebsch-Gordon coefficients (look them up in tables)

Note symmetry properties

eigenvectors or \( J = J_A + J_B \) symmetric under \( A \leftrightarrow B \)

\( J = J_A + J_B - 1 \) antisymmetric under \( A \leftrightarrow B \)

etc.

Graphical representation

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-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
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\( \downarrow m \)
Back to (particle) physics

Strong interaction \[ \mathcal{S} \]

\[ q_i \rightarrow q_i \]

\[ q_i q_i' \] coupling flavor-blind, i.e. does not depend on what quark it is, just depends on color.

To the extent \( m_u \approx m_d \), strong interaction invariant under exchange of \( u \) and \( d \).

Also independent of rotations between \( u \) and \( d \) states.

\[
\begin{pmatrix}
|u\rangle \\
|d\rangle
\end{pmatrix}
\rightarrow
U(x)\begin{pmatrix}
|u\rangle \\
|d\rangle
\end{pmatrix}
\]

\[ U(x) = e^{i\frac{K}{2}} \]

\[ K_i = K_i^\dagger \] generators

This is a rotation not in real space but in flavor space.

\( U(x) \) is \( 2 \times 2 \), can be complex.

4 complex elements \( \Rightarrow \) 8 real parameters.

\( U^TU = 1 \) imposes 4 constraints \( \Rightarrow \) 8 - 4 = 4 parameters.

\( \Rightarrow \) 4 generators - What are they?

Any hermitian matrix, can be written as linear combination of the following 4 matrices.
\[ K_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad K_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad K_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad K_4 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

This is the group of unitary transformations in 2D \[ U(2) \]

Transformation generated by \( K_4 \) is trivial

\[
\begin{pmatrix} 1 u \rangle \\ 1 d \rangle \end{pmatrix} = e^{i \alpha K_4} \begin{pmatrix} 1 u \rangle \\ 1 d \rangle \end{pmatrix} = e^{i \alpha} \begin{pmatrix} 1 u \rangle \\ 1 d \rangle \end{pmatrix} \quad \text{(just a phase shift)}
\]

This is a good symmetry but we do not much care about it.

Consider subgroup obtained from \( U(2) \) by removing the \( K_4 \) from the set of generators

This group is called \( SU(2) \)

Easy to see that \[ [K_i, K_j] = i \varepsilon_{ijk} K_k \]

Some Lie Algebra \( \mathfrak{su}(2) \) as for rotation group \[ [J_i, J_j] = i \varepsilon_{ijk} K_k \]

\( SU(2) \) = lowest dimensional representation of the rotation group.

Note \( K_i = \frac{1}{2} \sigma_i \quad \sigma_i \equiv \text{Pauli matrices} \)

Can think of \( 1u \rangle \) and \( 1d \rangle \) as being \( 1q \rangle \) state with additional quantum number \( \pm \frac{1}{2} \)

\( 1q + \frac{1}{2} \rangle = 1u \rangle \quad 1q - \frac{1}{2} \rangle = 1d \rangle \)

New quantum number: \( \boxed{\text{ISOSPIN}} \)

It has the same properties as \( \text{SPIN} \)

Combine Isospin like we combine angular momentum
Isospin doublet \((u,d)\) \(I = \frac{1}{2}\)

Like spin doublet \((\uparrow, \downarrow)\) \(S = \frac{1}{2}\)

Original isospin doublet introduced by Heisenberg \((p, n)\) \(I = \frac{1}{2}\)

Decay Good because \(m_p \neq m_n\)

Modern picture \(p \sim uud\)
\(n \sim udd\)

Combine three quarks \(qqq\) -
\[\begin{array}{ccc}
2 \otimes 2 \otimes 2 \\
(2S+1) & (2S+1) & (2S+1)
\end{array}\] (like adding angular momenta)

What is the total \(I\)

\[
\begin{align*}
(2 \otimes 2) \otimes 2 &= (1 \oplus 3) \otimes 2 = 2 \oplus 2 \oplus 4 \\
I = 0 & \quad I = 1 \\
I = \frac{1}{2} & \quad I = \frac{3}{2}
\end{align*}
\]

The \(p\) and \(n\) are combination of the two \(I = \frac{1}{2}\) representations. Consistent with Heisenberg \((p, n)\) \(I = \frac{1}{2}\)

\(p \sim uud\) \(I_3 = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = +\frac{1}{2}\)

\(n \sim udd\) \(I_3 = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}\)
What about the $I = \frac{3}{2}$?

These are the $\Delta^+ \Delta^+ \Delta^0 \Delta^-$.

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

$\text{nnn} \quad \text{nnp} \quad \text{udp} \quad \text{ddd}$