Schrödinger equation \( i\hbar \frac{d\psi}{dt} = H\psi \)

F: some observable - F is an operator

\( \langle F \rangle \): expectation value of F \( \langle F \rangle = \int d^3x \psi^* F \psi \)

When is F conserved?

F conserved means \( \frac{d\langle F \rangle}{dt} = 0 \)

\( \frac{d\langle F \rangle}{dt} = \frac{d}{dt} \int d^3x \psi^* F \psi = \int d^3x \frac{d\psi^*}{dt} F \psi + \int d^3x \psi^* F \frac{d\psi}{dt} \)

Take complex conjugate of Schrödinger equation

\( -i\hbar \frac{d\psi^*}{dt} = (H \psi)^* = \psi^* H \)

Use this relation into equation for \( \frac{d\langle F \rangle}{dt} \)

\( \frac{d\langle F \rangle}{dt} = \int d^3x \left( -\frac{i}{\hbar} \right) \psi^* H F \psi + \int d^3x \left( \frac{1}{i\hbar} \right) \psi^* F H \psi \)

\( \frac{d\langle F \rangle}{dt} = \frac{i}{\hbar} \int d^3x \psi^* (HF-FH) \psi = \frac{i}{\hbar} \int d^3x \psi^* [HF] \psi \)

If \( [HF] = 0 \) then \( \frac{d\langle F \rangle}{dt} = 0 \), F conserved
Then, the eigenfunctions of $H$ can be chosen so that they are also eigenfunctions of $F$

$$H\psi = E\psi$$
$$F\psi = \tilde{F}\psi$$

\[\text{Symmetry transformation}\]

$$\psi \rightarrow \psi', \text{ with } \psi'(\vec{x}, t) = U\psi(\vec{x}, t)$$

\[\text{Transformation operator}\]

Conservation of probability

$$\int d^3x \, \psi^* \psi = \int d^3x \, \tilde{\psi}^* \tilde{\psi} = \int d^3x \, (U\psi)^* U\psi = \int d^3x \, \psi^* U^* U \psi$$

Therefore

$U$ must be unitary $U^* U = 1$

$U$ is a "good" symmetry if $U\psi = \psi'$ satisfies Schrödinger equation

\[i\hbar \frac{d}{dt} \psi' = H \psi'\]

\[i\hbar \frac{d}{dt} (U \psi) = H U \psi\]

If $U$ is time independent

\[i\hbar \frac{d}{dt} U \psi = H U \psi\]
\[ i \hbar \frac{d\psi}{dt} = U^* H U \psi \]

\[ H = U^{-1} H U = U^+ H U \]

\[ U H = H U \]

\[ [u H] = 0 \]

\[ \uparrow \text{For a "good" symmetry} \]

**Recap**

- **Operator** \( F \), observable \( \Rightarrow \) \( \text{Even} \quad F = F^+ \)

  \[ \langle F \rangle \text{ conserved} \Rightarrow [H, F] = 0 \]

- **Operator** \( U \), unitary \( U^* U = 1 \)

  Symmetry transformation \( \psi' = U \psi \)

  Good symmetry : \( [H, U] = 0 \)

  (Obviously, if \( U = U^+ \) then there will be an observable associated with it, and the observable will be conserved)

**Two types of transformations**

- **Continuous**
- **Non-continuous**

**Definition**: continuous transformations connect smoothly to the unit operator. They are characterized by a set of continuous parameters.
Example

3D Rotation. Continuous. Parameters: 3 Euler angles
Parity \( \vec{x} \rightarrow -\vec{x}' \) - Not continuous

Parity operator: \( P \)
Apply twice: \( \vec{x} \rightarrow -\vec{x} \rightarrow \vec{x}' \), \( P^2 = 1 \)

So \( P^2 = 1 \) and \( PP^T = 1 \) \( \Rightarrow \) \( P = P^+ \) \( \Rightarrow \) observable called "parity" of the system associated with it.

In general, often, non-continuous transformations have hermitian operators. This is not generally the case for continuous continuous transformations.

Write that.

Write \( U = e^{i\alpha G} \)

\[ U\psi = e^{i\alpha G}\psi = (1 + i\alpha G + (i\alpha G)^2/2 + \ldots) \psi \]

\( \alpha \): real parameter
\( G \): generator of \( U \)
In general \( U \neq U^+ \)

But \( UU^+ = 1 \)
\[ i\alpha G - i\alpha G^T \]
\[ e \quad e^{i\alpha G^T} = 1 \]
gives \( G = G^T \)
The generator $G$ is hermitian $\implies$ associated with an observable.

Now consider infinitesimal transformation, i.e. let $\alpha$ be very small

$$U = e^{i\alpha G} \approx 1 + i\alpha G$$

If $U$ is a good symmetry transformation, then

$$[U, H] = 0$$
$$UH - HU = 0$$
$$(1 + i\alpha G)H - H(1 + i\alpha G) = 0$$
$$H + i\alpha GH - H - i\alpha HG = 0$$
$$i\alpha (GH - HG) = 0$$
$$i\alpha [G, H] = 0$$

Since $\alpha$ is arbitrary $[G, H] = 0$

$\implies$ The observable associated with $G$ is conserved.

Example

Consider 1D system

wavefunction $\psi(x)$

Consider translation by $\Delta x$

$\psi'(x) = \psi(x + \Delta x)$ or $\psi'(x) = U(\Delta x)\psi(x)$

Expand in Taylor series, keep only first order term, i.e. look at infinitesimal transformation.
\[ \Psi'(x) = \Psi(x) + \Delta x \frac{\partial \Psi}{\partial x} = U(\Delta x) \Psi(x) \]

\[ \Rightarrow U(\Delta x) = 1 + \Delta x \frac{\partial}{\partial x} = 1 + \Delta x G \]

\[ G = \frac{\partial}{\partial x} \]

But \( x \)-momentum operator \( \hat{P}_x = -i\hbar \frac{\partial}{\partial x} \)

So \( G = \frac{\partial}{\partial x} = \frac{i}{\hbar} \hat{P}_x \)

\( \hat{P}_x \) is the generator of translations in the \( x \)-direction

If translations in the \( x \)-directions are a "good" symmetries, then \( P_x \) is conserved

Similarly, rotation about an axis, say the \( z \)-axis, can be shown to be generated by

\[ G = \frac{\partial}{\partial \phi} \]

in cylindrical (or spherical) coordinates

Then consider angular momentum operator, \( z \)-component

\[ \hat{J}_z = (\hat{r} \times \hat{P})_z = -i\hbar (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) = -i\hbar \frac{\partial}{\partial \phi} \]

So \( G = \frac{i}{\hbar} \hat{J}_z \)

\( \hat{J}_z \) \( (\hat{J}_x, \hat{J}_y) \) is the generator of rotations about the \( z \) \((x, y)\) axis
If rotations are "good" symmetries, $\mathcal{J}$ conserved