The uncertainty principle (or, that particles are waves) keeps the world from collapsing.

\[ H = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2m} - \frac{e^2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \]

kinetic energy

of m = electron mass

assuming proton \( \infty \) mass

coulomb

attraction;

\( x, y, z \) are

coordinate of \( e^- \)

w/r to proton.

\[ \langle H \rangle = \frac{\langle \dot{x}^2 \rangle + \langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle}{2m} - e^2 \langle \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \rangle \]

seek to minimize \( \langle H \rangle \); that will be the ground state. We'll find that the closer you squish the electron into the proton, thereby reducing the potential energy contribution, the ensuing localization pushes the kinetic energy up (via uncertainty principle).

Lowest energy is result of a compromise between the two terms.
To make progress, need some relationships:

\[
\langle p_x^2 \rangle = \langle p_x \rangle^2 + (\Delta p_x)^2
\]

\[\text{variance.}\]

recall: \[\begin{aligned}
(\Delta p_x)^2 &= \langle \psi \left( p_x - \langle p_x \rangle \right)^2 \rangle \\
&= \langle \psi p_x^2 \psi \rangle - 2 \langle \psi p_x \psi \rangle \langle p_x \rangle + \langle \psi \langle p_x \rangle^2 \rangle \\
&= \langle p_x^2 \rangle - 2 \langle p_x \rangle^2 + \langle p_x \rangle^2 \langle \psi \psi \rangle \\
&= \langle p_x^2 \rangle - \langle p_x \rangle^2 + \langle p_x \rangle^2 \langle \psi \psi \rangle
\end{aligned}\]

\[\begin{aligned}
(\Delta p_x)^2 &= \langle p_x^2 \rangle - \langle p_x \rangle^2 \\
\langle p_x^2 \rangle &= \langle p_x \rangle^2 + (\Delta p_x)^2 \quad \text{same for } y, z
\end{aligned}\]

\[\Rightarrow \text{minimizing } H \text{ suggests } \langle p_x \rangle = \langle p_y \rangle = \langle p_z \rangle = 0 \]

then

\[H = \frac{(\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2}{2m} - e^2 \left( \frac{1}{(x^2 + y^2 + z^2)^2} \right)\]

"Handwaving"

\[\left( \frac{1}{(x^2 + y^2 + z^2)^2} \right)^{1/2} \approx \frac{1}{\langle x^2 + y^2 + z^2 \rangle^{1/2}} \approx \frac{1}{\langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle}\]
\[ \langle x^2 \rangle = \langle x \rangle^2 + (\Delta x)^2 \] et cetera

\[ \langle H \rangle \approx \frac{(\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2}{2m} - \frac{e^2}{\left[ (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \right]^{1/2}} \]

set to zero, since origin is arbitrary.

want to minimize... most minimize:

\[
\begin{align*}
(\Delta p_x)^2 &= (\Delta p_y)^2 = (\Delta p_z)^2 \\
(\Delta x)^2 &= (\Delta y)^2 = (\Delta z)^2
\end{align*}
\]

True, but not proven here.

\[ \langle H \rangle \approx \frac{3(\Delta p_x)^2}{2m} - \frac{e^2}{\sqrt{3} \Delta x} \]

\[ \Delta p_x \Delta x > \frac{\hbar}{2} \]

\[ (\Delta p_x)^2 > \left( \frac{\hbar}{2\Delta x} \right)^2 \]

\[ \langle H \rangle \gg \frac{3\hbar^2}{8m\Delta x^2} - \frac{e^2}{\sqrt{3} \Delta x} \]

kinetic potential

dominant as dominant as

\[ \Delta x \to 0 \]

\[ \Delta x \to \infty \]
\[ \langle H \rangle \sim -\frac{2}{q} \frac{me^4}{\hbar^2} \quad \text{exact} \quad -\frac{1}{2} \frac{me^4}{\hbar^2} \]

\[ \langle H \rangle \sim -\frac{3}{8} \frac{h^2}{m \Delta x^2} \quad \text{kinetic energy} \]

\[ \sim \frac{1.3}{me^2} \]

\[ \Delta x = \frac{3\sqrt{3}}{4} \frac{h^2}{me^2} \approx 1.3 \frac{h^2}{me^2} \]

\[ \frac{d}{d(\Delta x)} \langle H \rangle \sim -3 \frac{h^2}{4m(\Delta x)^3} + \frac{e^2}{\sqrt{3} \Delta x^2} = 0 \]

\[ \langle \Delta x \rangle = \frac{4}{3\sqrt{3}} \frac{me^2}{h^2} \]
Product Space

particle #1 \to \psi \rangle \text{ state vector}

particle #2 \to \phi \rangle \text{ state vector}

\psi \rangle \phi \rangle \text{ means: measure #1, get info described by } \psi \rangle

measure #2, get info described by \phi \rangle

Not all states in a product space are products! Key is linear superposition.

#1 \to \psi_a \rangle \quad \psi_b \rangle \quad \text{are 2 possible states of #1}

#2 \to \phi_a \rangle \quad \phi_b \rangle

then: \psi_a \rangle \phi_a \rangle + \psi_b \rangle \phi_b \rangle

is in the product space, but is not in a product. It is "entangled" though.

"Example"

\psi \rangle \chi \rangle = \chi \psi \rangle \text{ (eigenstate of particle #1's position)
\[ |x_2\rangle : \quad x_2|x_2\rangle = x_2 |x_2\rangle \]

**idea:** \( |x_1\rangle|x_2\rangle \) is in the **product space**

\[
\begin{align*}
x_1 [ |x_1\rangle|x_2\rangle ] &= x_1 |x_1\rangle|x_2\rangle \\
x_2 [ |x_1\rangle|x_2\rangle ] &= x_2 |x_1\rangle|x_2\rangle
\end{align*}
\]

\[ |x_1\rangle|x_2\rangle : \quad |x_1\rangle \text{ is in one space (} \#1 \text{)} \]
\[ |x_2\rangle \text{ is in another (} \#2 \text{)} \]

\[ x_1 : \text{ in } \#1 \text{’s space, } x_1 \]
\[ x_2 : \text{ in } \#2 \text{’s space, } x_2 \]

\[ \text{also: } \quad x^{(1)}_1 \otimes x^{(2)}_2 = x^{(1)}_1 \otimes x^{(2)}_2 \] (formally)

\[ \text{means paste together} \]
\[ |x_1\rangle \otimes |x_2\rangle = |x_1\rangle|x_2\rangle \text{ or } |x_1, x_2\rangle \]

**Physics**

\[
\frac{1}{\sqrt{2}} \left( |x_1' x_2'\rangle + |x_1'' x_2''\rangle \right) \quad \text{if } x_1' \neq x_1'' \quad \text{and} \quad x_2' \neq x_2''
\]

make wave function: project into eigenbras of position:

\[
\frac{1}{\sqrt{2}} \langle x_1 x_2 | \left[ |x_1' x_2'\rangle + |x_1'' x_2''\rangle \right] = (\text{distributes})
\]

\[
= \frac{1}{\sqrt{2}} \langle x_1 x_2 | x_1' x_2'\rangle + \frac{1}{\sqrt{2}} \langle x_1 x_2 | x_1'' x_2''\rangle
\]
$x_2$ passes through $x'_1$ but not $x''_2$, etc.

$$= \frac{1}{r_2} \langle x_1 | x'_1 \rangle \langle x_2 | x'_2 \rangle + \frac{1}{r_2} \langle x_1 | x''_1 \rangle \langle x_2 | x''_2 \rangle \delta(x_1 - x'_1) \delta(x_2 - x'_2) \delta(x_1 - x''_1) \delta(x_2 - x''_2)$$

Wavefunction:

$$= \frac{1}{\sqrt{2}} \delta(x_1 - x'_1) \delta(x_2 - x'_2) + \frac{1}{\sqrt{2}} \delta(x_1 - x''_1) \delta(x_2 - x''_2)$$

Measure $x_1 \rightarrow$ get $x'_1 \Rightarrow$ know $x''_2$ without measurement (never see $x'_1, x''_2$).

Measure $x_1 \rightarrow$ get $x''_1 \Rightarrow$ know $x_2 = x''_2$ without measurement (never see $x''_1, x'_2$).

"Entanglement" non-product state in product space.
\[ \left[ \frac{1}{\sqrt{2}} (|x_1\rangle + e^{i\theta_1} |x_1\rangle) \right] \left[ \frac{1}{\sqrt{2}} (|x_2\rangle + e^{i\theta_2} |x_2\rangle) \right] \]

\[ \theta_1, \theta_2 = \text{real #s} \]

this is a product state, in product space.

Matrices That Represent Product Operators

Best to work through example:
Exercise 10.1.2 p. 251:

Particle #1: two states, call eigenstates
\[ |+\rangle, |-> \]

Particle #2: two states in different space, also call
\[ |+\rangle, |-> \] (just mentally)

4 product states span the product space

\[ |+\rangle |+\rangle \text{ or } |+\rangle \otimes |+\rangle \text{ or } |+\rangle |+\rangle \]
\[ |+\rangle |-> \text{ or } |+\rangle \otimes |-> \text{ or } |+\rangle |-> \]
\[ |-> |+\rangle \text{ or } |-> \otimes |+\rangle \text{ or } |-> |+\rangle \]
\[ |-> |-> \text{ or } |-> \otimes |-> \text{ or } |-> |-> \]