Here is a review of some (but not all) of the topics you should know for the midterm. These are things I think are important to know. I haven’t seen the test, so there are probably some things on it that I don’t cover here. Hopefully this covers most of them.

- **Vector Spaces**
  Review properties on Shankar page 2
  Closure under multiplication: If $|u\rangle$ and $|v\rangle \in V$, then $a|u\rangle + b|v\rangle \in V$ for any $a, b$. Note when $b = 0$ this takes care of scalar multiplication also.
  Inverses, identity, etc.

- **Linear independence**
  A set of vectors $\{|v_i\rangle\}$ is linearly independent if $a|v_1\rangle + b|v_2\rangle + \cdots = 0$ has only one solution: $a = b = \cdots = 0$.

- **Gram-Schmidt procedure**
  If you have a set of linearly independent vectors $|I\rangle, |II\rangle, \ldots$ you can always construct an orthonormal set of vectors as follows:

  $|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I \rangle}}$  
  $|2\rangle = \frac{|II\rangle - |1\rangle\langle 1|II \rangle}{\text{normalization constant}}$  
  $|3\rangle = \frac{|III\rangle - |1\rangle\langle 1|III \rangle - |2\rangle\langle 2|III \rangle}{\text{normalization constant}}$  

  The normalization constants are chosen so that $\langle 2|2 \rangle = 1, \langle 3|3 \rangle = 1, \ldots$.

- **Basis**
  A basis of a vector space $V$ is a set of vectors $\{|v_i\rangle\}$. 
Any vector $|u\rangle \in V$ can be written in terms of these vectors: $|u\rangle = a|v_1\rangle + b|v_2\rangle + \ldots$ always has $a, b, \ldots$ so that the equation is satisfied.

- **Orthonormal (ON) basis**
  An ON basis is one for which $\langle v_i|v_j\rangle = \delta_{ij}$.

- **Decomposition of unity**
  If \{|$v_i\rangle$\} is an ON basis, then $\sum_i |v_i\rangle\langle v_i| = \mathbb{I}$.

- **Linear Operators**
  Linear operators have $\Omega(a|u\rangle + b|v\rangle) = a\Omega|u\rangle + b\Omega|v\rangle$.

- **Operator Inverses**
  The inverse of the product of operators is given by the inverses of those operators in reverse order: $(\Omega\Lambda)^{-1} = \Lambda^{-1}\Omega^{-1}$.

- **Commutators**
  The commutator of two matrices is written $[A, B] \equiv AB - BA$. The anticommutator is written $\{A, B\} = [A, B]^+ = AB + BA$.

- **Hermitian, Unitary, etc.**
  An operator $\Lambda$ is Hermitian if $\Lambda = \Lambda^\dagger$. It is unitary if $\Lambda^{-1} = \Lambda^\dagger$ or equivalently $\Lambda\Lambda^\dagger = \mathbb{I}$.
  An operator is anti-Hermitian if $\Lambda = -\Lambda^\dagger$. An operator is anti-unitary if, among other things, $\Lambda(a|u\rangle) = a^*\Lambda|u\rangle$. Anti-Hermitian and anti-unitary operators won’t show up often (if at all) in this class—in fact, I can think of only one anti-unitary operator that comes up in physics.
• Projection operators
Defining equation: $\tilde{P}^2 = \tilde{P}$. $Tr \tilde{P}$ = dimensionality of subspace onto which $\tilde{P}$ projects. Example: $\mathbb{I}^2 = \mathbb{I}$. The trace of an operator is the sum of the diagonal elements of its matrix representation. In $N$ dimensions, the identity operator is a $N \times N$ matrix with $N$ 1’s on the diagonal, so $Tr \mathbb{I} = N$.

• Matrix elements
Inserting a decomposition of unity twice,
$$\Omega_{ij} = \langle i | \Omega | j \rangle$$
$$\Omega = \sum_{ij} |i\rangle \Omega_{ij} \langle j|$$

For a vector, the components are given by
$$v_i = \langle i | v \rangle$$
$$|v\rangle = \sum_i |i\rangle \langle i | v \rangle = \sum_i v_i |i\rangle$$

• Change of basis
A change of basis from one ON basis (the “unprimed” basis $\{|i\rangle\}$) to another basis (the “primed” basis $\{|i'\rangle\}$) transforms operators and vectors as follows (inserting decompositions of $\mathbb{I}$),
$$\langle i'| A | j' \rangle = \sum_{ij} \langle i'|i \rangle \langle i | A | j \rangle \langle j | j' \rangle$$
$$\langle i'|v\rangle = \sum_i \langle i'|i \rangle \langle i | v \rangle$$
Note that $U_{jj'}$ is a matrix for which the $j^{th}$ basis vector goes in $j^{th}$ column.

- **Eigenvectors, eigenvalues**
  If
  \[ A|v\rangle = a|v\rangle \quad |v\rangle \neq 0 \]
  then $|v\rangle$ is an eigenvector of $A$ with eigenvalue $a$.

- **Determining eigenvalues**
  Solve the equation
  \[ \det(A - aI) = 0 \]
  where $A$ is a matrix representation of $A$. The left hand side ends up being a polynomial called the “characteristic polynomial” of the operator, and the equation is called the “characteristic equation” of the operator. For an $N$ by $N$ matrix $A$, the polynomial is an $N^{th}$-order polynomial, and so the equation has $N$ solutions. They need not be distinct – one or more of the eigenvalues can be the same number. If that happens, that eigenvalue is called “degenerate.”

- **Eigenvectors**
  Once you have the eigenvalues, solve
  \[ A \begin{bmatrix} \alpha \\ \beta \\ \vdots \end{bmatrix} = a \begin{bmatrix} \alpha \\ \beta \\ \vdots \end{bmatrix} \]
  for each eigenvalue to get the associated eigenvector $(\alpha, \beta, \ldots)$. If the eigenvalue is nondegenerate, you’ll have $N$ unknowns $\alpha, \beta, \ldots$ and $N - 1$ equations
one-parameter family of eigenvectors. Impose normalization condition $\alpha^*\alpha + \beta^*\beta \ldots = 1$ to fix the final free parameter.

If the eigenvalue is $m$-fold degenerate ($m$ of the eigenvalues are the same) then you get $N$ free parameters and $N-(1+m)$ equations, and thus an $m$-parameter family of eigenvectors. Example: suppose you get the eigenvector

$$|v\rangle = \begin{bmatrix} \alpha \\ \beta \\ -\beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

You can split it up into two or more “basis” vectors that “span the degenerate subspace”–in the above example, any eigenvector of that eigenvalue can be written as a linear combination of the two vectors with combination coefficients $\alpha$ and $\beta$.

- **Diagonalization**

  If $\{|i\rangle\}$ are the normalized eigenvectors of $A$, you can represent $A$ in that “eigenbasis”, and if the eigenvectors are normalized, the new matrix representation will be a diagonal matrix with the eigenvalues as the diagonal elements. As discussed for changes of basis, $U$ is constructed

  $$U = \begin{bmatrix} |1\rangle & |2\rangle & \ldots \end{bmatrix}$$

  If the eigenvectors are not normalized, you’ll still get
a diagonal matrix, but the diagonal elements will not be the eigenvalues of \( A \).

- Simultaneous Diagonalization
  Suppose we have a matrix \( B \) that commutes with \( A \): 
  \[ [A, B] = 0. \]
  Then the ON basis that diagonalizes \( A \) is the same ON basis that diagonalizes \( B \)—the eigenvectors of \( B \) are the same as the (orthonormal) eigenvectors of \( A \), but with different eigenvalues \( (B|i⟩ = b|i⟩) \).
  They diagonalize \( B \) into a matrix with \( B \)'s eigenvalues on the diagonal.

  One of the reasons one cares about this is illustrated as follows. Suppose you have a 1000 by 1000 matrix \( B \). The characteristic equation is a 1000th-order polynomial. For 2nd order polynomials, the quadratic equation can solve the characteristic equation; for 3rd and 4th order polynomials we also have equations. But for higher-order polynomials there is no general way of finding the roots, and so finding the eigenvalues would be very hard. But, if you can find an \( A \) that commutes with \( B \), you can find the eigenvalues and eigenvectors of \( A \) instead of solving the characteristic equation for \( B \). If you can find an \( A \) for which diagonalization is very easy, then all you have to do is matrix multiplication to diagonalize \( B \) and find its eigenvalues. It saves a lot of work.

- Delta functions
  Suppose you have an interval \( \gamma \) (e.g. \( \gamma = (-\infty, \infty) \)).
Then the defining equation of a delta function is

$$\int_{\gamma} f(x)\delta(x)dx = f(0)$$

if $0 \in \gamma$ and the result is zero if zero is not in the interval. Also, integrating by substituting $u = g(x)$,

$$\int_{\gamma} f(x)\delta(g(x))dx = \int_{\gamma} f(x(u))\delta(u)\frac{du}{dg(x(u))} = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

where $x_i$ is a solution of $g(x_i) = 0$ and $x_i \in \gamma$. Why the absolute value sign is required is a homework problem for Monday. Finally, integrating by parts,

$$\int_{\gamma} f(x)\frac{d\delta(x)}{dx}dx = f(x)\delta(x)\bigg|_{\partial\gamma} - \int_{\gamma} \frac{df}{dx}\delta(x)dx$$

where $\partial\gamma$ is the boundary of the interval $\gamma$ (e.g. if $\gamma = (-1, 1)$, $f(x)\delta(x)|_{\partial\gamma} = f(x)\delta(x)|_1^{-1}$). Since $\delta(x)$ is zero everywhere except $x = 0$, the first term is zero as long as $0 \in \gamma$ (and not on the boundary) and so

$$\int_{\gamma} f(x)\frac{d\delta(x)}{dx}dx = - \int_{\gamma} \frac{df}{dx}\delta(x)dx$$