Physics 115A Midterm

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Monday, Feb. 10, 2003

Closed Book; no calculators. For full credit, show your work and make your reasoning clear to the graders.

The ‘boldface’ notation below is used for operators; thus, $\Omega$ is an abstract operator. In class we put a ‘twiddle’ under the $\Omega$ to denote that it was an operator. The symbol $\doteq$ means ‘is represented by’.

The quadratic formula for the roots to the equation $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. (25 pts) Two kets have unit length: $|V_1\rangle$, and $|V_2\rangle$, so $\langle V_1|V_1\rangle = \langle V_2|V_2\rangle = 1$; these two kets are never equal, that is, $|V_1\rangle \neq |V_2\rangle$. The two projection operators are $P_1 = |V_1\rangle\langle V_1|$ and $P_2 = |V_2\rangle\langle V_2|$.

   (a) Suppose $|V_1\rangle$ and $|V_2\rangle$ are represented in an orthonormal basis by:

   $$|V_1\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |V_2\rangle \doteq \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{bmatrix}.$$

   i. Find the matrices that represent $P_1$ and $P_2$.

   ii. Use the matrix representations to find the matrix that represents the commutator $[P_1, P_2]$.

   (b) In general, what conditions on $|V_1\rangle$ and $|V_2\rangle$ will guarantee that the commutator $[P_1, P_2] = 0$?

2. (20 pts) Consider the linear operator $\Omega$ which operates on abstract vectors in a space of dimension 2, and which is represented in one particular basis by the matrix:

   $$\Omega \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

   (a) Is $\Omega$ Hermitian?

   (b) Is $\Omega$ unitary?

   (c) What are the eigenvalues of $\Omega$?

   (d) What are the representations of the normalized eigenvectors of $\Omega$?

Over...
3. (40 pts) The linear operators $\Omega$ and $\Lambda$ operate on abstract vectors in a space of dimension 3, and in one particular orthonormal basis they are represented by the matrix:

$$
\Omega = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
\frac{2}{3}b & b & \frac{1}{3}b \\
\frac{1}{3}b & b & b \\
\frac{1}{3}b & b & \frac{2}{3}b
\end{bmatrix}
$$

where $b$ is a non-zero real number.

(a) Do $\Omega$ and $\Lambda$ commute?

(b) Is $\Omega$ unitary?

(c) One eigenket of $\Omega$, $|\omega_3\rangle$, has an obvious representation in this basis, namely:

$$
|\omega_3\rangle = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix};
$$

what is the eigenvalue $\omega_3$ that corresponds to this eigenket?

(d) Find the other two eigenvalues of $\Omega$; call $\omega_1$ the smaller of the two, and $\omega_2$ the larger of the two.

(e) Find the unitary matrix that transforms the representation of $\Omega$ given above into the diagonal form:

$$
\Omega \doteq \begin{bmatrix}
\omega_1 & 0 & 0 \\
0 & \omega_2 & 0 \\
0 & 0 & \omega_3
\end{bmatrix}
$$

(f) Apply the same unitary transformation to $\Lambda$.

(g) What are the eigenvalues of $\Lambda$?

4. (15 pts) Numerically evaluate the integral:

$$
\int_{-\infty}^{\infty} \delta(4x - 2) \left[ \frac{1}{2}x^2 - 1 \right] dx
$$