# Visualizing Electromagnetic Knots

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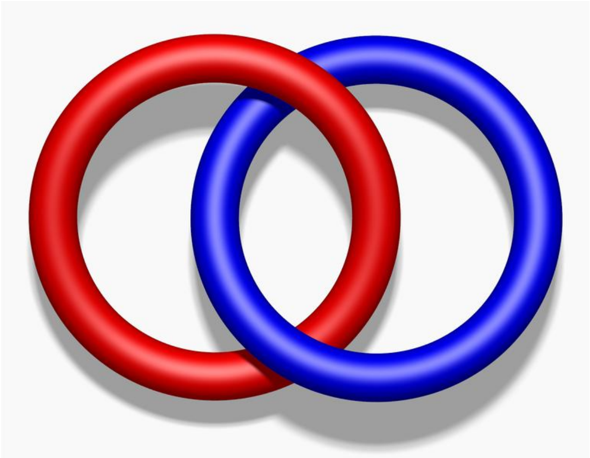
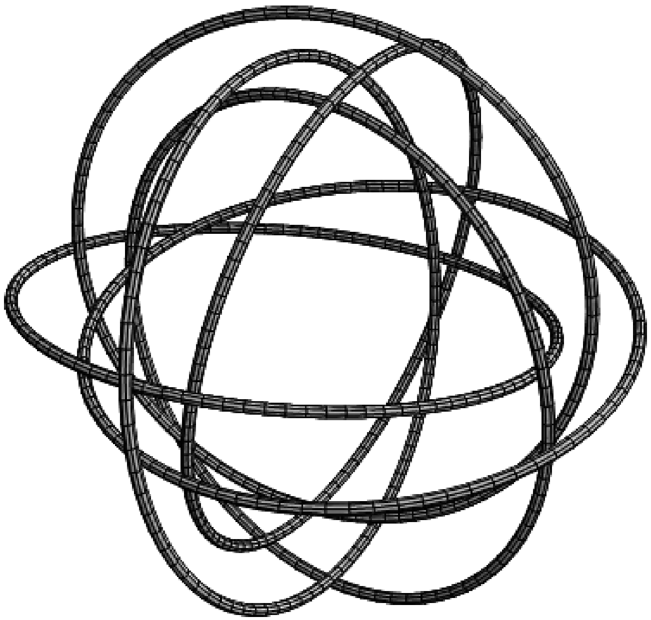
# Introduction

- Maxwell's equations admit strange solutions wherein any two field lines are linked (topologically). These are known as **electromagnetic knots**.
- **Rañada** et al. recently introduced a **topological theory of electromagnetism** based upon these knots.



Linked circles

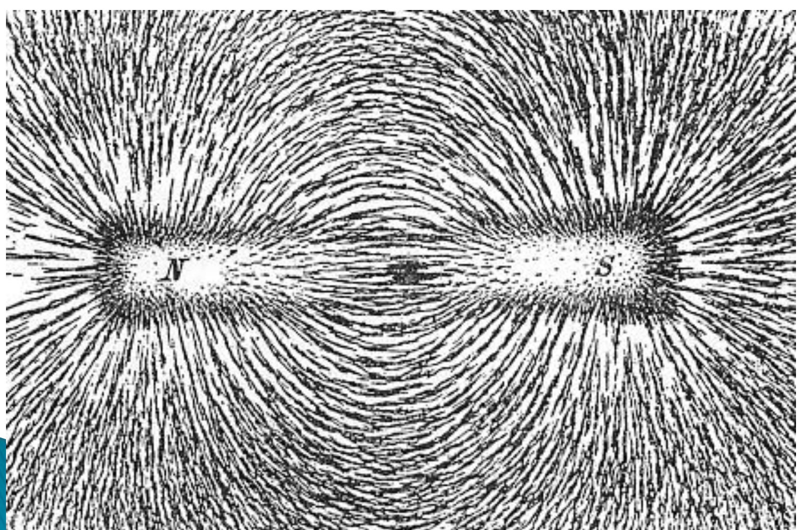


Early diagram of EM knot

# Motivation

- Faraday understood EM fields in terms of “lines of force”, now known as **field lines**.
- Faraday’s intuitive understanding was later encoded mathematically in Maxwell’s equations.
- Intuition and visualization breed scientific progress!

Faraday




Maxwell

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

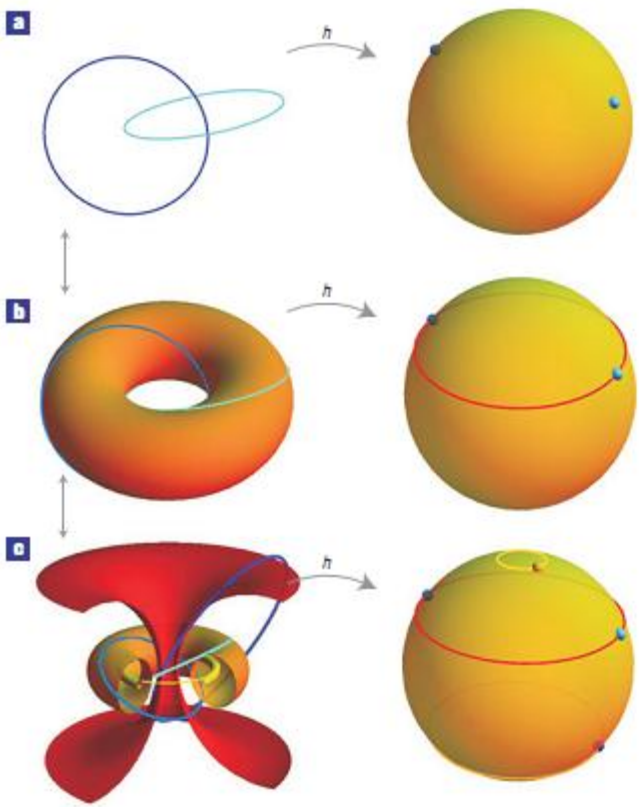
$$\nabla \cdot \mathbf{B} = 0$$

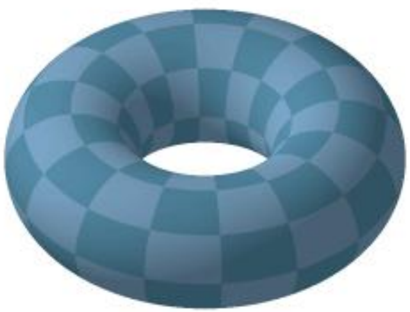
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

# Geometry

- Field line structure based on the **Hopf map**
  - Complex scalar field in space
  - Level curves are distinct circles
  - Surfaces of constant modulus are nested tori



# Electromagnetic Knots

- Original construction of the fields cast in terms of differential geometry
- Initial solutions based on the Hopf map. Time-evolved solutions found by Fourier analysis
- The expressions are given by

$$\boldsymbol{B} = \frac{1}{2\pi i}\frac{\nabla\phi \times \nabla\bar{\phi}}{(1+\bar{\phi}\phi)^2}, \qquad \phi = \frac{(ax - tz) + i(ay + t(a-1))}{(az + tx) + i(a(a-1) - ty)},$$

$$\boldsymbol{E} = \frac{1}{2\pi i}\frac{\nabla\theta \times \nabla\bar{\theta}}{(1+\bar{\theta}\theta)^2}, \qquad \theta = \frac{\big(ay + t(a-1)\big) + i(ax + tz)}{(az + tx) + i(a(a-1) - ty)}$$

where $a = \frac{1}{2}(r^2 - t^2 + 1)$.

# Time Evolution of Field Lines

- We would like to visualize the time-evolved field lines while maintaining the topological structure induced by the Hopf map.
  - This is not always possible and requires the fields to behave in a certain manner.

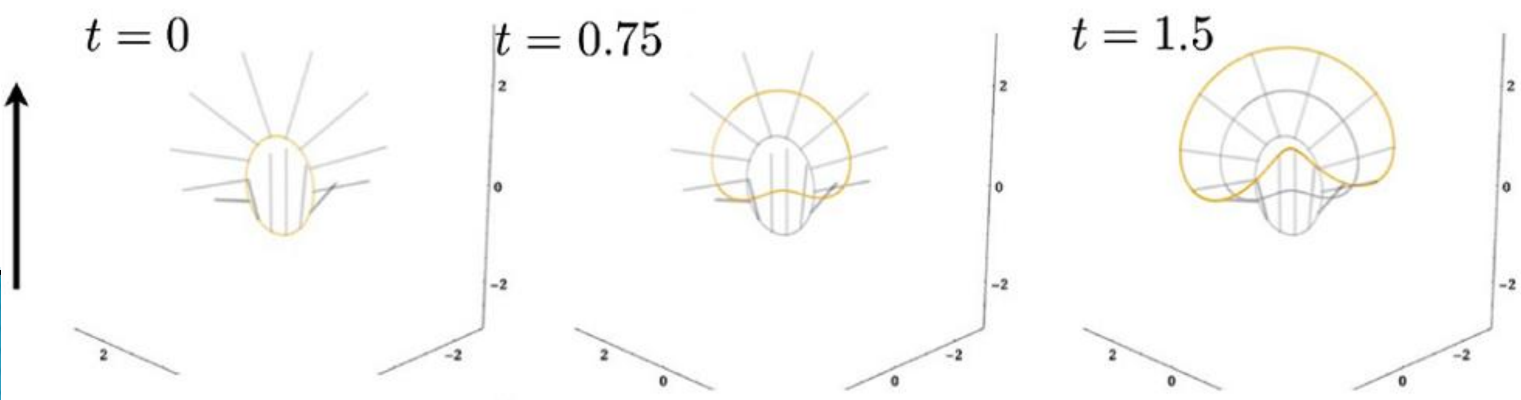
- The electric and magnetic fields satisfy the **"frozen field" condition**
  - There exists a **velocity field** along which the field lines deform, maintaining their identity as such
  - Only possible because $E \cdot B = 0$ at all times.

# Velocity Field

- The corresponding velocity field is given by

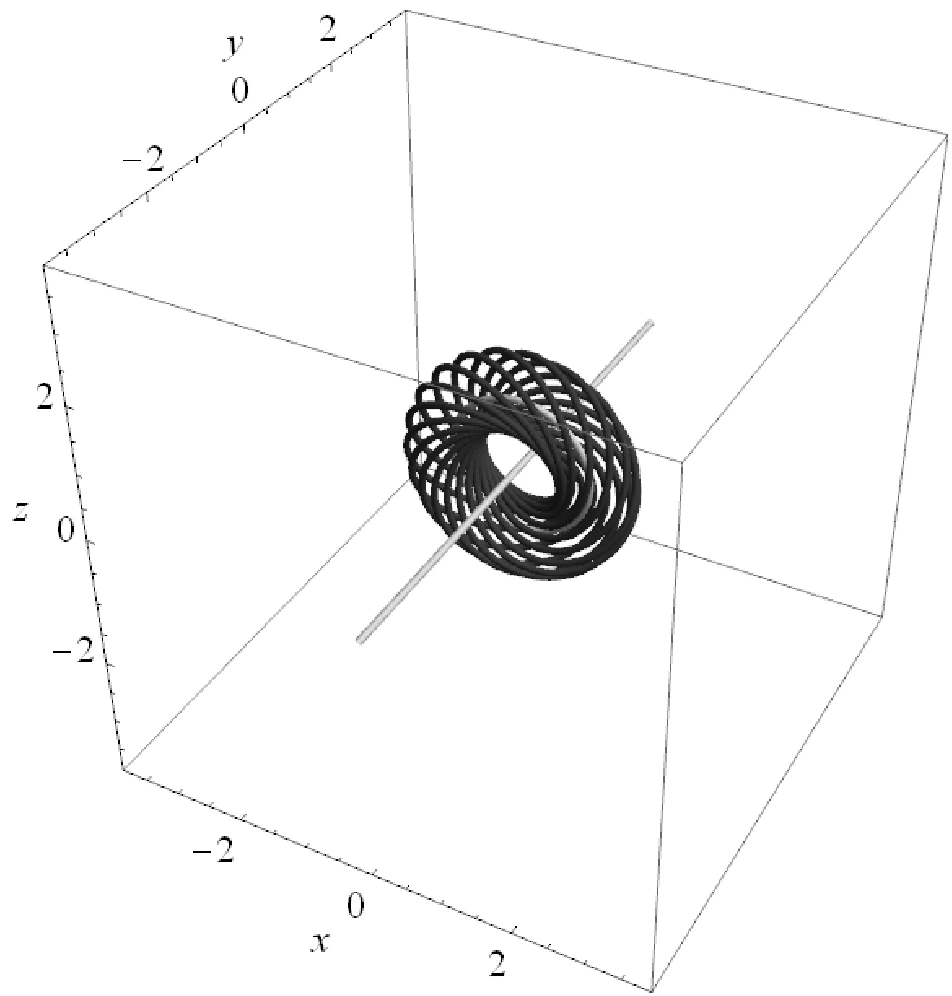
$$v = \frac{E \times B}{E \cdot E} = \frac{E \times B}{B \cdot B}$$

- Some surprising facts
  - The velocity field depends functionally on $z + t$
  - Any single element of a field line travels along a straight line at the speed of light.



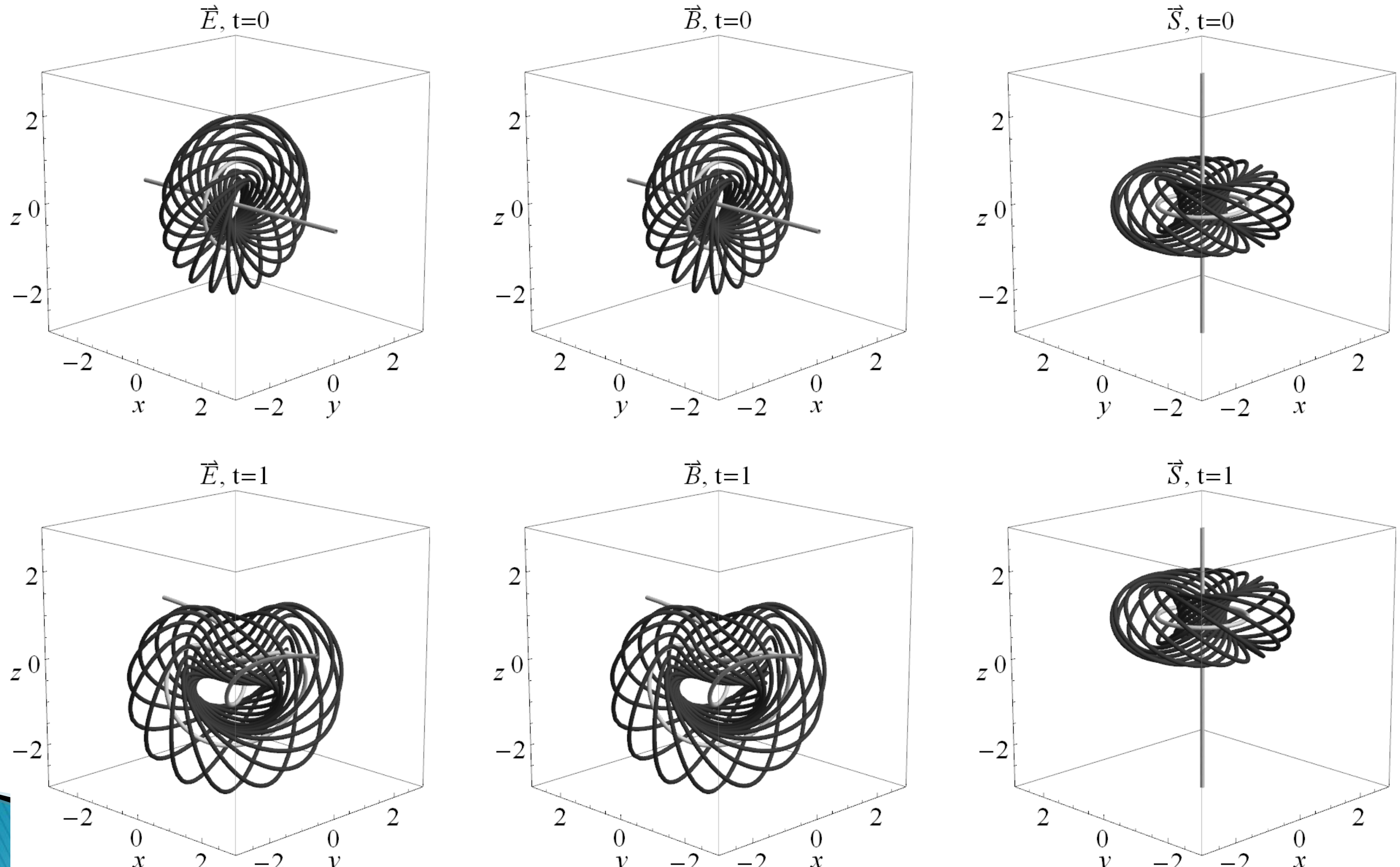
# Parametric Visualization

- In order to parametrize the field lines, we solve for the inverse of the Hopf map parametrically.
- Time-evolved field lines are parametrized by deformation via the velocity field described previously.
- The result is a simple parametric description of the field lines.



# Figure

- These images are based on a different construction of the knots involving complex fields.
- Will be used in a colleague's paper



# Prospects

- Gain a greater understanding of the nature of electromagnetism and EM knots.
- Apply our understanding to related areas, such as complex fields, complexified spacetime, and twistor theory.