Parameterization and Zeros in the Rate

Harry Nelson

December 2, 1997

1. **Mixing:** Neglecting CP violation, normalizing the right-sign amplitude to 1, the amplitude for wrong sign through mixing is $-iMt$, where M is the mixing amplitude and t the proper time since birth of the D^0:

$$M = \frac{1}{2}\left[\Delta M - \frac{i}{2}\Delta\Gamma\right] = \frac{\overline{\Gamma}}{2}\left[x - iy\right] = -i\frac{\overline{\Gamma}}{2}e^{-i\phi_{\rm SW}}\sqrt{x^2+y^2}$$

where

$$\overline{\Gamma} = \frac{1}{\tau_{D^0}}\ ,\ x = \frac{\Delta M}{\overline{\Gamma}}\ ,\ y = \frac{\Delta\Gamma}{2\overline{\Gamma}}\ ,\ \phi_{\rm SW} = \tan^{-1}\left[-2\frac{\Delta M}{\Delta\Gamma} = -\frac{x}{y}\right]$$

The mean D^0 life is τ_{D^0}. Transitions from $D^0 \to \overline{D^0}$ through virtual intermediate states contribute to x, which is real when CP violation is neglected in the transitions to or from the intermediate states. The parameter x can have any value from $-\infty$ to ∞. Transitions from $D^0 \to \overline{D^0}$ through real intermediate states contribute to y: if final states with CP=1(CP=-1) dominate the D^0 width, y=1(-1). By convention, we take y to be real, even in presence of CP violation. If $|y| << |x|$, as is likely in the D^0 system, then $y>0 (y<0)$ would correspond to $\cos\phi_{\rm SW}=1 (\cos\phi_{\rm SW}=-1)$. Presence of CP violation can render x complex, and when $|{\rm Im}(x)| \approx x$ and $|x| \approx |y|$, the utility of the above relations is destroyed.

In the limit that mixing is small, the 'mixing rate' $R_{\rm mixing}$, is:

$$R_{\rm mixing} = \frac{\int_0^\infty dt e^{-\overline{\Gamma}t}|Mt|^2}{\int_0^\infty dt e^{-\overline{\Gamma}t}} = 2\frac{|M|^2}{\overline{\Gamma}^2} = \frac{1}{2}\left[x^2+y^2\right]$$

So, the mixing amplitude becomes:

$$-iMt = -\sqrt{\frac{R_{\rm mixing}}{2}}e^{-i\phi_{\rm SW}}\tau$$

where $\tau = \overline{\Gamma}t$ is the proper time measured in units of mean D^0 lifetimes.

2. **Direct Decay:** The right and wrong sign decay amplitudes are:

$$A = \langle K^-\pi^+|\mathcal{H}|D^0\rangle \propto V_{cs}V_{ud} = \cos^2\theta_C\ \ ,\ \ B = \langle K^+\pi^-|\mathcal{H}|D^0\rangle \propto V_{cd}V_{us} = -\sin^2\theta_C$$

It is the 'GIM' mechanism that describes destructive interference between an amplitude that involves CKM element V_{us} and another that involves V_{cd}; thus the GIM mechanism describes the minus sign above and in:

$$\frac{B}{\overline{A}} = \rho e^{-i\delta_s}[-\tan^2\theta_C] \quad , \quad R_{\rm DCSD} = \left|\frac{B}{\overline{A}}\right|$$

where the real ρ describes an adjustment due to final state rescattering, as does the real phase δ_s, and θ_C is the Cabibbo angle. For $D^0 \to K^-\pi^+$, it is expected that $\rho > 1$. All CP violations have been neglected: *direct* CP violation would render $|A| \neq |\overline{A}|$ and/or $|B| \neq |\overline{B}|$, while CP violation in *interference* would keep the magnitudes of A and $\overline{A}$ (and/or B and $\overline{B}$) equal, but would cause them to differ by a complex phase.

3. **Time-dependent rate, including interference:** Combining the two above, and normalizing to $|A| = 1$, and neglecting all CP violation, the total rate of wrong-sign $D^0 \to K^-\pi^+$ decays, $w(\tau)$ is:

$$\begin{aligned} w(\tau) &= \left| -\sqrt{R_{\rm DCSD}}e^{-i\delta_s} - \sqrt{\frac{R_{\rm mixing}}{2}}e^{-i\phi_{\rm SW}}\tau \right|^2 \\ &= R_{\rm DCSD} + \sqrt{2R_{\rm DCSD}R_{\rm mixing}}\cos(\delta_s + \phi_{\rm SW})\tau + \frac{1}{2}R_{\rm mixing}\tau^2 \end{aligned} \tag{1}$$

The terms of order τ^0, τ^1, and τ^2 are often called the 'direct decay' or 'DCSD' term, the 'interference' term, and the 'mixing term,' respectively.

4. **Zeros:** The zeros in Eqn. 1 occur when:

$$\tau = 2\sqrt{\frac{R_{\rm DCSD}}{x^2+y^2}}e^{\pm i(\delta_s+\phi_{\rm SW})} \; . \tag{2}$$

There is zero for τ real and positive when $x=0$, $y<0$ and $\delta_s=0$. For $\sqrt{R_{\rm DCSD}} = \tan^2\theta_C = 0.0513$, $x=0$, and $y=-0.075$, Eqn. 2 predicts that the zero is at:

$$\tau = 2\times\frac{0.0513}{0.075} \approx 1.37(= 1.36930)$$

This zero is shown in Fig. 1.

5. **Higher Orders:** The inclusion of terms higher order than second to Eqn. 1 changes the time-dependent rate, and also shifts the location of the zeros. The question is, by how much?

The three terms in Eqn. 1 become, when all orders of τ are included:

$$\begin{aligned} R_{\rm DCSD} &\Longrightarrow R_{\rm DCSD}\times\frac{1}{2}[\cosh y\tau + \cos x\tau] \\ \sqrt{2R_{\rm DCSD}R_{\rm mixing}}\cos(\delta_s+\phi_{\rm SW})\tau &\Longrightarrow \sqrt{R_{\rm DCSD}}\,[\cos\delta_s\sinh y\tau + \sin\delta_s\sin x\tau] \\ \frac{1}{2}R_{\rm mixing}\tau^2 &\Longrightarrow \frac{1}{2}[\cosh y\tau - \cos x\tau] \end{aligned} \tag{3}$$

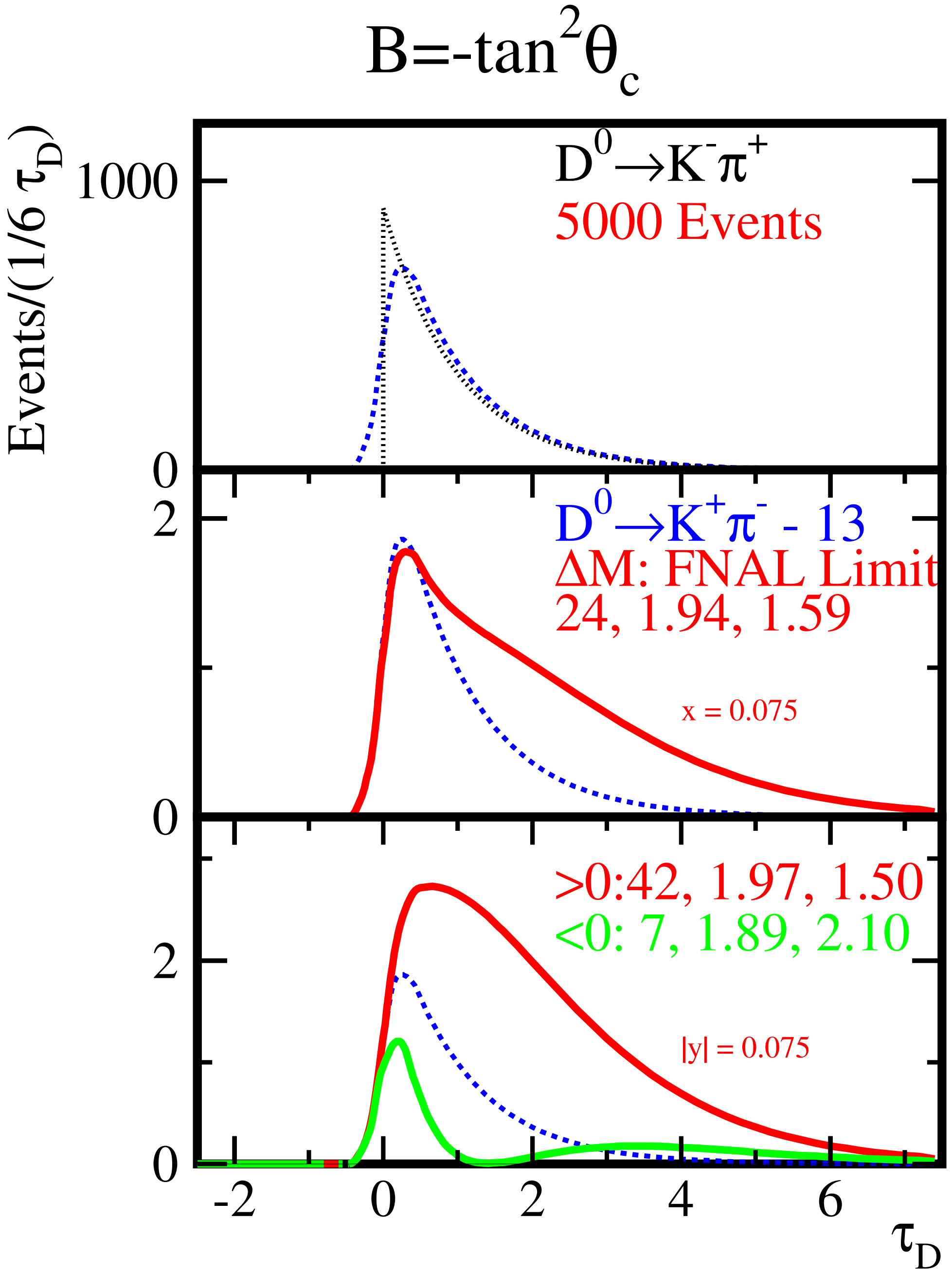

Figure 1: Interference patterns for various mixing assumptions. The top plot shows the pure exponential, and the effect of a slight smearing in blue: the area corresponds to 5000 right sign events. For the lower two plots, the direct wrong-sign decay amplitude is fixed to $-\tan^2\theta_C$, where θ_C is the Cabibbo angle. The integers give the number of wrong-sign events for the various cases, and the real numbers are the mean lifetime and the standard deviation, respectively. The middle case shows pure dispersive or ΔM mixing with $x = 0.075$; the bottom case the two extremes of $y = \pm 0.075$. For $y = -0.075$, there is a zero, as predicted by Eqn. 2, at $\tau_D = 1.37$.

For $\sqrt{R_{\mathrm{DCSD}}} = \tan^2\theta_C = 0.0513$, $x = 0$, and $y = -0.075$, inclusion of all orders in τ shifts the location of the zero from $\tau = 1.36930$ to $\tau = 1.37051$, about 0.09%. This shift is not significant.

The error in the time-dependent wrong-sign rate made by keeping only second-order terms as in Eqn. 1 is summarized in Fig. 2. Normalized to the right-sign direct amplitude, the absolute error never exceeds 1 part per million. The *relative* error is typically 1%, and grows as τ increases, because the long lived component of the D^0 decays exponentially.

However, near the zero, the relative error blows up, because the exact time-dependent rate vanishes at a τ that is slightly different than the τ at which the second-order approximation of Eqn. 1 vanishes. This behavior is not so important, because the rate is so small in this region.

For $\sqrt{R_{\mathrm{DCSD}}} = \tan^2\theta_C = 0.0513$, $x = 0$, and $y = 0.075$, there is no zero, and the second order approximation and the exact expression differ in absolute terms (with normalization to the right-sign direct-decay term) by less than about 10 parts per million; the relative difference never becomes blows up, but grows monotonically from 0 up to about 5% by $\tau = 10$. Switching $x = 0.075$ and $y = 0$, the absolute difference stays below 3 parts per million, and the relative difference replays a pattern similar to the $x = 0$, $y = 0.075$ case.

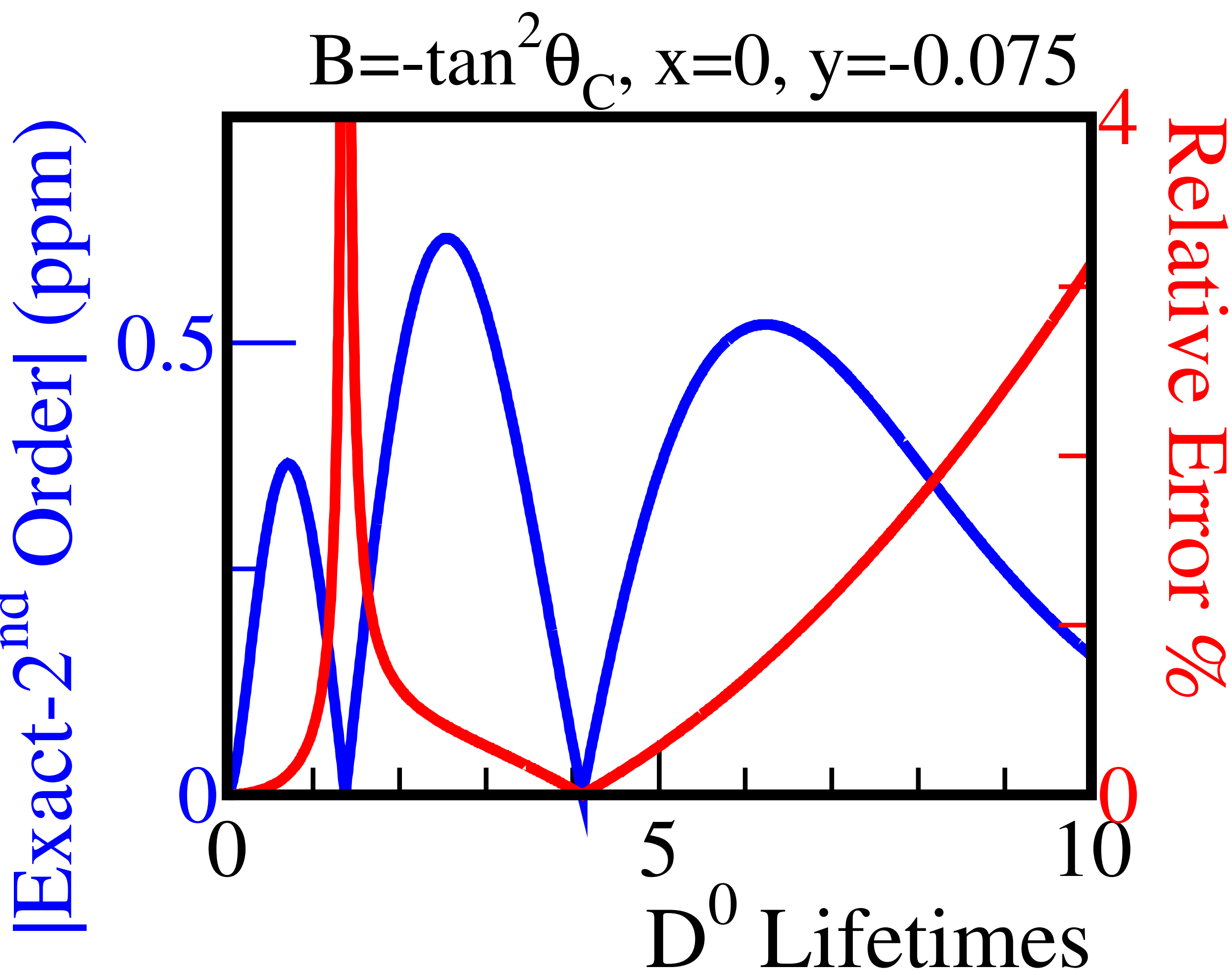

Figure 2: Error on the time-dependent wrong-sign decay rate made through use of the second-order approximation, Eqn. 1, of the exact expression, Eqn. 3. The direct right-sign decay amplitude is taken as 1, and the direct wrong-sign decay amplitude $R_{\rm DCSD} = |B| = |-\tan^2\theta_C| = |-0.0513|$. Dispersive mixing is absent, described by $x = 0$, while absorptive mixing is present, described by $y = \Delta\Gamma/(2\overline{\Gamma}) = -0.075$. The curve with three peaks is the magnitude of the error in parts per million, and should be read with the vertical scale on the left. The curve with the peak near 1.37 D^0 lifetimes is the relative error, an should be read with the vertical scale on the right, which is in percent. The peak in this curve near 1.37 D^0 lifetimes occurs because Eqn. 1 and Eqn. 3 hit zero at slightly different numbers of D^0 lifetimes. This curve rises as the number of lifetimes increases, because the long-lived component of the D^0 decays exponentially.