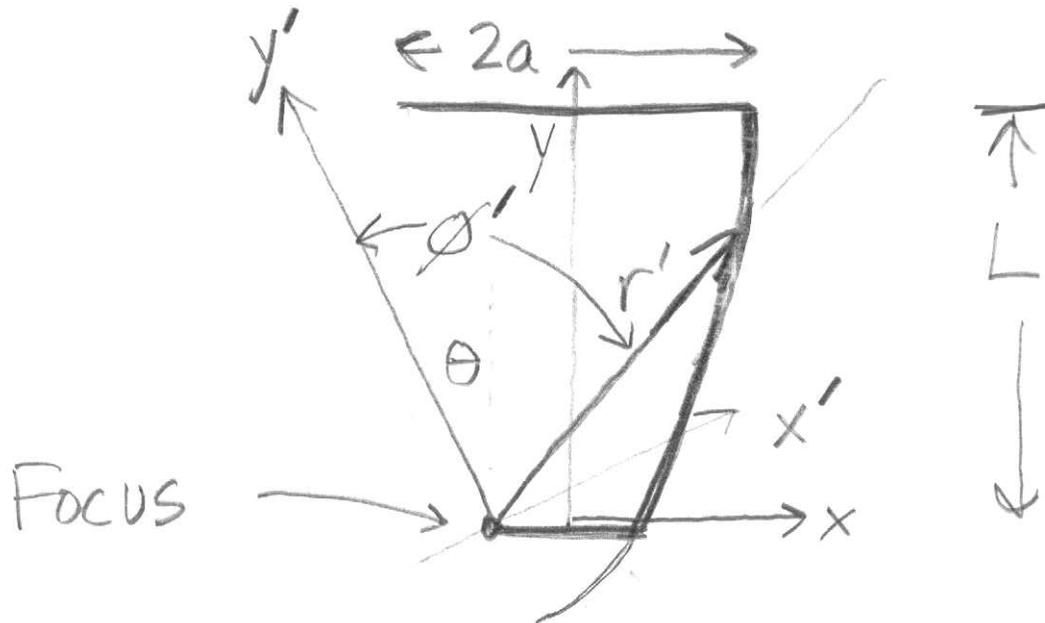


Computation of the (saturated) Winston Cone



Using the focus as the origin and the x' - y' coordinates with polar coordinates $r' + \phi'$, ϕ' is with respect to the y' axis

Parabola:

$$r' = \frac{r_0}{1 - \cos \phi'} = \frac{r_0}{2 \sin^2(\phi'/2)}$$

r_0 is the only parameter that characterizes the parabola. This is a consequence of choosing the focus as the origin

r_0 can be determined from the requirement that the parabola pass through the far side of the output surface

$$2b = \frac{r_0}{1 - \cos(\theta + \frac{\pi}{2})} = \frac{r_0}{1 + \sin\theta}$$

$$\boxed{r_0 = 2b(1 + \sin\theta)}$$

The parabola in x-y system

$$x = x' \cos \theta - y' \sin \theta - b$$

$$y = y' \cos \theta + x' \sin \theta$$

$$x' = r' \sin \phi'$$

$$y' = r' \cos \phi'$$

$$x = r' (\cos \theta \sin \phi' - \sin \theta \cos \phi') - b$$

$$x = r' \sin(\phi' - \theta) - b$$

$$y = r' \cos(\phi' - \theta)$$

$$x = \frac{r_0}{2} \frac{\sin(\phi' - \theta)}{\sin^2(\phi'/2)} - b \quad y = \frac{r_0}{2} \frac{\cos(\phi' - \theta)}{\sin^2(\phi'/2)}$$

The concentration power of the Winston depends on how large x can grow. Since the parabola's parameters are fixed, we can compute this.

$$X_{\max} \rightarrow \frac{dx}{d\phi'} = 0$$

$$\frac{dx}{d\phi'} = \frac{r_0}{2} \left[\frac{\cos(\phi' - \theta)}{\sin^2(\phi'/2)} - \frac{2 \sin(\phi' - \theta) \cos(\phi'/2) \cdot \frac{1}{2}}{\sin^3(\phi'/2)} \right] = 0$$

$$\cos(\phi' - \theta) - \sin(\phi' - \theta) \frac{\cos(\phi'/2)}{\sin(\phi'/2)} = 0$$

$$\text{or } \frac{\cos(\phi' - \theta)}{\sin(\phi' - \theta)} = \frac{\cos(\phi'/2)}{\sin(\phi'/2)}$$

$$\text{so } \phi' - \theta = \phi'/2$$

$$\phi'/2 = \theta$$

$$\boxed{\phi' = 2\theta}$$

This proves
picture on page 4

$$X_{\max} = \frac{r_0}{2} \frac{\sin(\phi' - \theta)}{\sin^2(\phi'/2)} - b$$

$$= \frac{1}{2} \cdot 2b(1 + \sin\theta) \frac{\sin\theta}{\sin^2\theta} - b$$

$$= \frac{b}{\sin\theta} + b - b = X_{\max} = \frac{b}{\sin\theta}$$

proves boxed
equation on
p. 3

So, the maximum radius at the cone is... $a = \frac{b}{\sin \theta}$ or $b = a \sin \theta$

The length L of the cone is then:

$$y(x_{\max}) = \frac{r_0}{2} \frac{\cos(\theta' - \theta)}{\sin^2 \theta'/2}$$

$$= \frac{1}{2} 2b(1 + \sin \theta) \frac{\cos \theta}{\sin^2 \theta}$$

$$L = y(x_{\max}) = b(1 + \sin \theta) \frac{\cos \theta}{\sin^2 \theta}$$

$$= a(1 + \sin \theta) \frac{\cos \theta}{\sin \theta} = a \left(1 + \frac{1}{\sin \theta}\right) \cos \theta$$

This proves the equation given on page 4.

$x(y)$ for the Winston Cone

Idea: specify y , then compute x for the Winston Cone (saturated)

Given y , compute ϕ' :

$$y = \frac{r_0}{2} \frac{\cos(\phi' - \theta)}{\sin^2 \phi'/2}$$

$$\xi \equiv \frac{y}{r_0} = \frac{\cos(\phi' - \theta)}{1 - \cos \phi'}$$

$$\xi - \xi \cos \phi' = \cos \phi' \cos \theta + \sin \phi' \sin \theta$$

$$\xi - (\xi + \cos \theta) \cos \phi' = \sin \theta \sin \phi'$$

$$\begin{aligned} \xi^2 - 2\xi(\xi + \cos \theta) \cos \phi' + (\xi + \cos \theta)^2 \cos^2 \phi' &= \sin^2 \theta \sin^2 \phi' \\ &= \sin^2 \theta (1 - \cos^2 \phi') \end{aligned}$$

$$\begin{aligned} (\xi^2 - \sin^2 \theta) - 2\xi(\xi + \cos \theta) \cos \phi' + \underbrace{[(\xi + \cos \theta)^2 + \sin^2 \theta]}_{\xi^2 + 2\xi \cos \theta + 1} \cos^2 \phi' &= 0 \end{aligned}$$

\Rightarrow

$$(1 + \xi^2 + 2\xi \cos \theta) \cos^2 \phi' - 2\xi(\xi + \cos \theta) \cos \phi' + (\xi^2 - \sin^2 \theta) = 0$$

Discriminant:

$$4\xi^2(\xi + \cos\theta)^2 - 4(1 + \xi^2 + 2\xi\cos\theta)(\xi^2 - \sin^2\theta)$$

$$= 4\xi^4 + 8\xi^3\cos\theta + 4\xi^2\cos^2\theta$$

$$- 4\xi^2 + 4\sin^2\theta - 4\xi^4 + 4\xi^2\sin^2\theta$$

$$- 8\xi^3\cos\theta + 8\xi\cos\theta\sin^2\theta$$

$$D^2 = 4\sin^2\theta(1 + 2\xi\cos\theta)$$

$$\cos\phi' = \frac{\xi(\xi + \cos\theta) \pm \sin\theta\sqrt{1 + 2\xi\cos\theta}}{(1 + \xi^2 + 2\xi\cos\theta)}$$

$$\xi \equiv \frac{y}{r_0}$$

when $D^2 = 0$... $1 + 2\xi\cos\theta = 0$

$$\xi = -\frac{1}{2\cos\theta}$$

$$1 + \xi^2 + 2\xi\cos\theta = 1 + \frac{1}{4\cos^2\theta} - 2\frac{1}{2\cos\theta}\cos\theta$$

$$= \frac{1}{4\cos^2\theta}$$

$$\cos\phi' = \frac{-\frac{1}{2\cos\theta}\left(-\frac{1}{2\cos\theta} + \cos\theta\right)}{\frac{1}{4\cos^2\theta}}$$

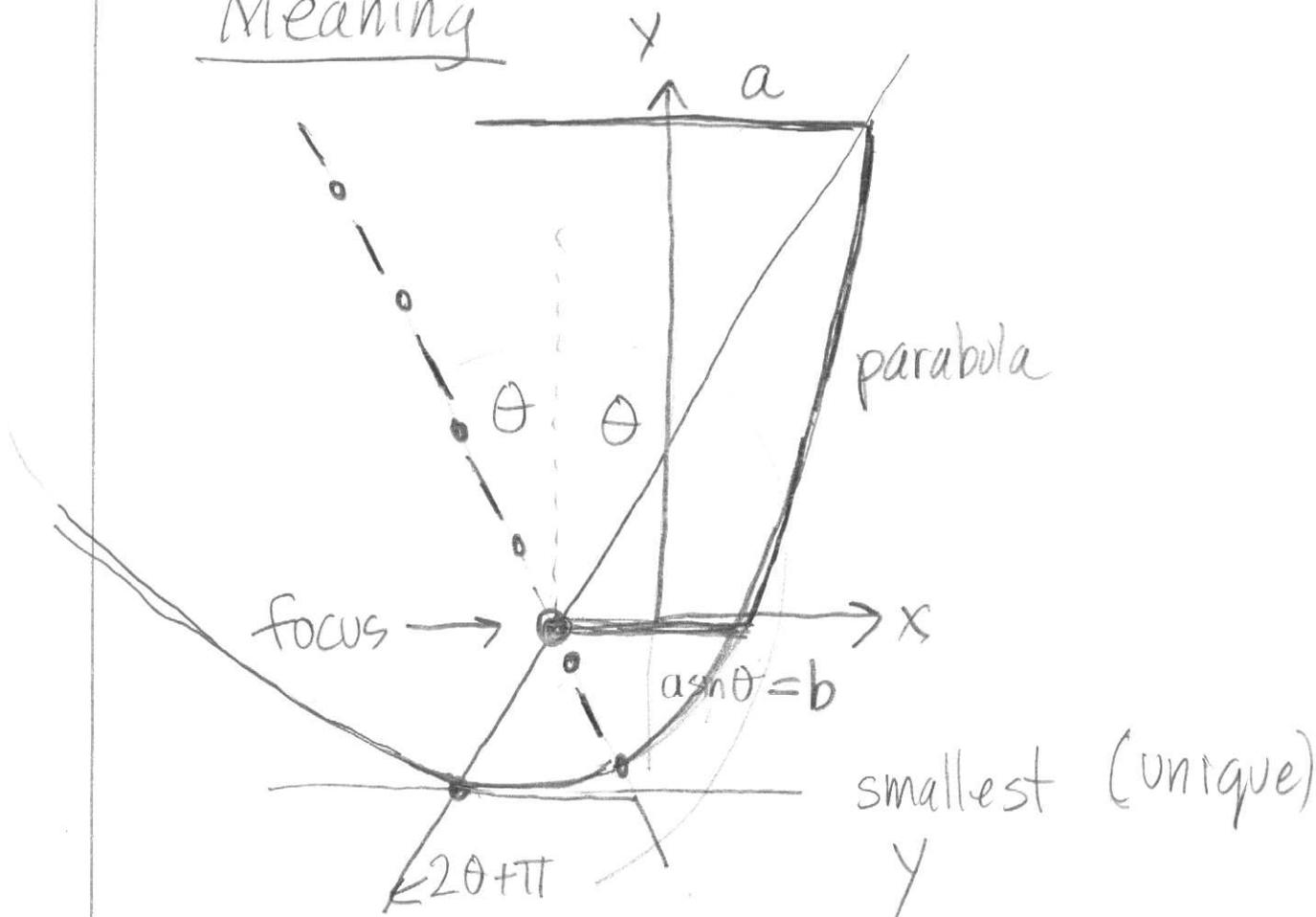
$$\begin{aligned}\cos \phi' &= -2 \frac{\cos^2 \theta}{\cos \theta} \left(-\frac{1}{2 \cos \theta} + \cos \theta \right) \\ &= -2 \cos \theta \left(-\frac{1}{2 \cos \theta} + \cos \theta \right) \\ &= -2 \left(-\frac{1}{2} + \cos^2 \theta \right)\end{aligned}$$

$$\begin{aligned}\cos \phi' &= 1 - 2 \cos^2 \theta = 1 - \cos^2 \theta - \cos^2 \theta \\ &= \sin^2 \theta - \cos^2 \theta\end{aligned}$$

$$\cos \phi' = -\cos 2\theta$$

$$\boxed{\phi' = 2\theta + \pi}$$

Meaning



To generate $x(y)$

- ① compute r_0 from $\theta \pm a$ (or b)
- ② Pick y
- ③ compute $\xi = \frac{y}{r_0}$
- ④ compute $\cos \phi'$ from $-$ branch of quadratic equation; get ϕ'
- ⑤ compute x, y from $r' = \frac{r_0}{1 - \cos \theta'}$ & equations on p, b