Fall 2004 Physics 3
Tu-Th Section

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Lecture 9: 21 Oct. 2004

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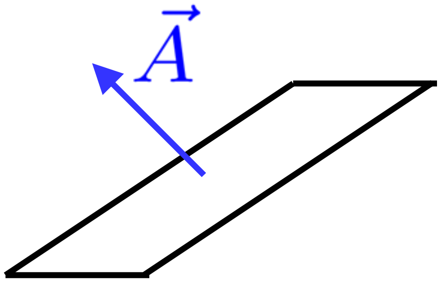
Last time: Gauss's Law

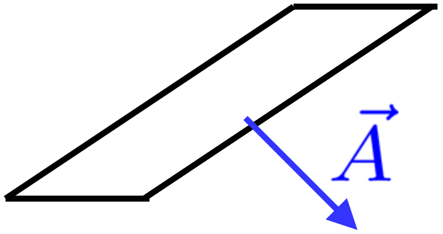
- To formulate Gauss's law, introduced a few new concepts
  - ➢ Vector Area
  - ➢ Electric Field Flux
- Let's review them

Vector Area

- A vector associated with a surface
- Magnitude of the vector = area of surface
- Direction of vector: perpendicular to surface



- Ambiguity: why not like this

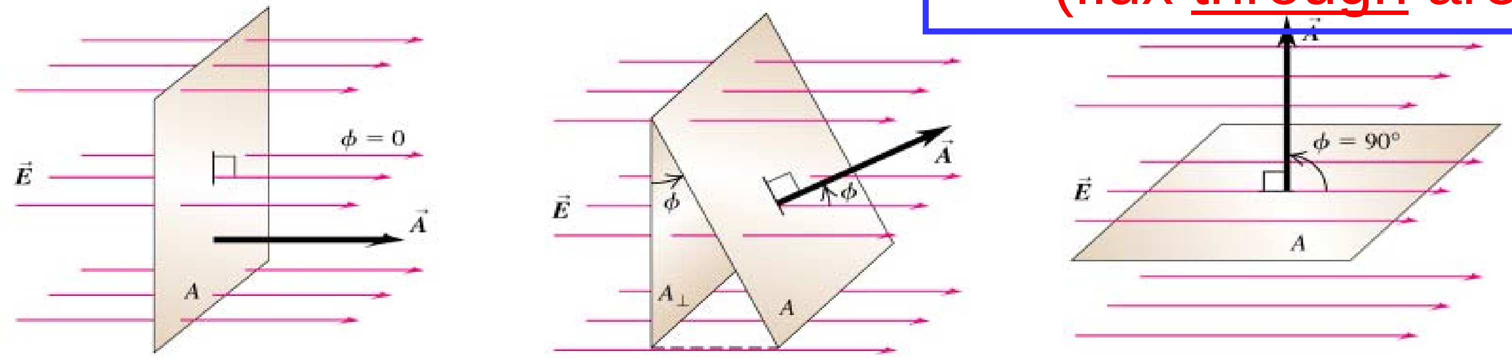


- Choice of direction is arbitrary
  - But you must specify it!

Electric Field Flux

- Definition: $\Phi_E = \vec{E} \cdot \vec{A}$

Flux always defined with respect to some area (flux <u>through</u> area)



$\Phi_E = EA$ $\Phi_E = EA\cos\phi$ $\Phi_E = 0$

- In general could have non-uniform E-field and non-flat surface. Then

$$\Phi_E = \int E \cos\phi \, dA = \int E_\perp \, dA = \int \vec{E} \cdot d\vec{A}$$

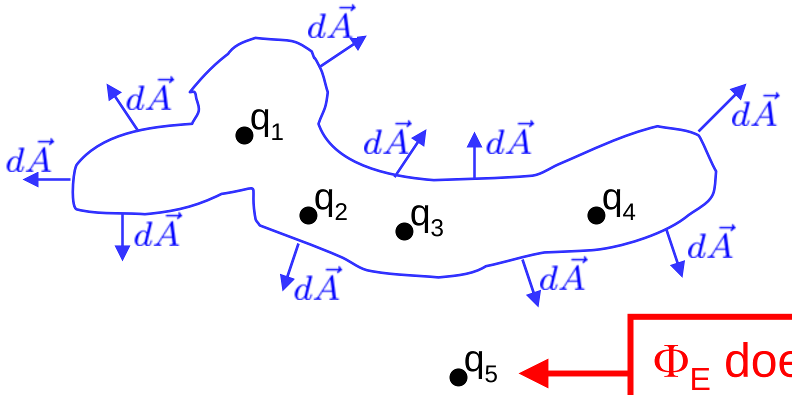
This is called a <u>surface integral</u>

Gauss's Law

- The electric field flux through a closed surface is proportional to the total charge enclosed by the surface

$$\Phi_E = \oint \vec{E} d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Note: $\oint$ means integral over a closed surface



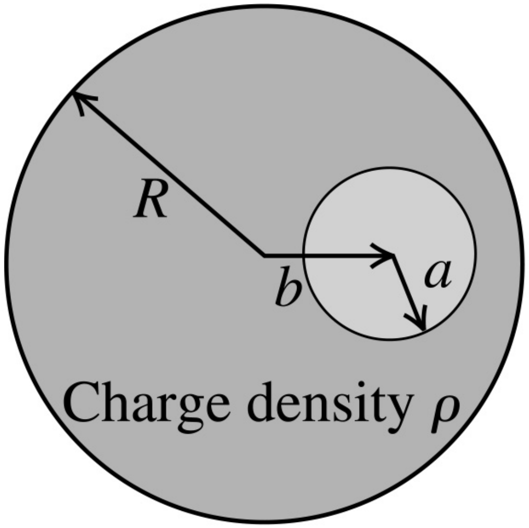
$d\vec{A}$ always points outward

$Q_{\text{enclosed}} = q_1 + q_2 + q_3 + q_4$

Φ_E does not depend on q_5 (outside surface)

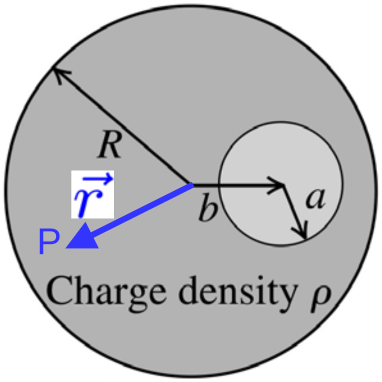
Another example (similar to Problem 22.61)

Insulating sphere of radius *R* charge density ρ.
The sphere has a hole at radius *b* of radius *a*.
Find the E field in the insulator and in the hole.



Trick: use principle of superposition:

1. solid sphere radius *R*, charge density ρ
2. solid sphere, radius *a*, charge density -ρ



Trick: use principle of superposition:
1. solid sphere radius *R*, charge density ρ
2. solid sphere, radius *a*, charge density -ρ

Use result of "example 2" from last lecture for field of 1:

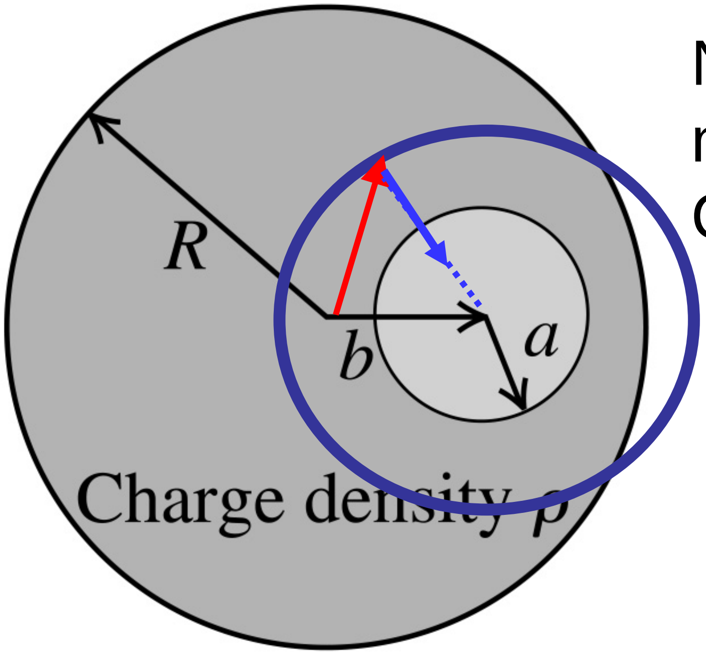
$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \vec{r}$$

Here I wrote it as a vector equation. The r-vector points from the center of the big sphere to the point at which we want E.

Careful: $\vec{r}$ here is **not** a constant vector. We want the field at some point P. The vector $\vec{r}$ is the vector that tells us where this point P actually is!

Recast this using ρ = (Q/V) and V = $4\pi R^3/3$

$$\vec{E}_1(\vec{r}) = \frac{\rho}{3\epsilon_0} \vec{r}$$



Now the field due to the fictitious negative charge density in the hole. Call this field E_2.

First, look outside the hole.

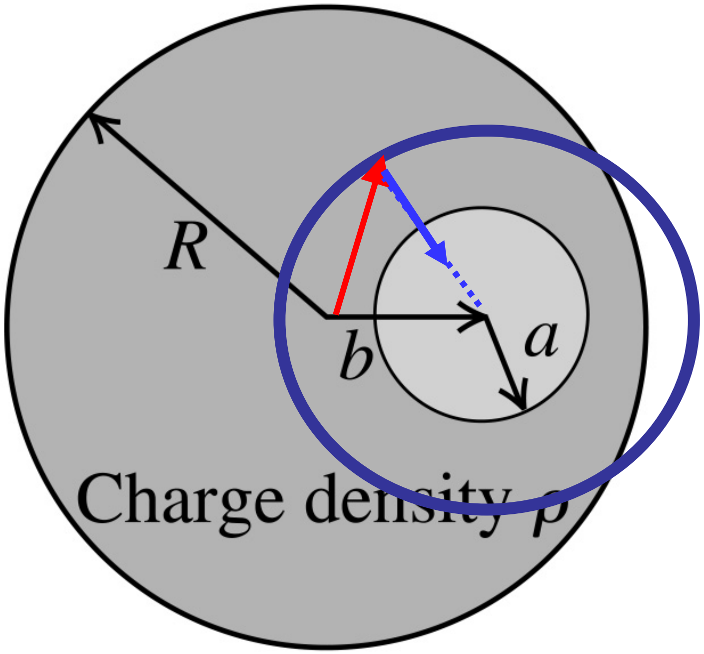
Draw imaginary (gaussian) sphere

The red vector is the position vector ($\vec{r}$) for a point on the gaussian sphere.

The blue vector is $\vec{E}_2$.

Let $\vec{d}$ be the vector that joins the point on gaussian sphere with center of spherical hole: $\vec{r}+\vec{d}=\vec{b}$.

$\vec{E}_2$ is in the direction of $\vec{d}=\vec{b}-\vec{r}$.



Red vector is $\vec{r}$.

Blue vector is $\vec{E}_2$.

$\vec{E}_2$ is in direction of $\vec{b}-\vec{r}$, which is also direction of $\vec{d}$.

Radius of gaussian sphere $|\vec{d}\,| = |\vec{b}-\vec{r}\,|$

Want E_2, electric field of sphere of radius a, charge density ρ

Using previous result, <u>outside sphere radius a</u>, we can pretend that all the charge is at the center

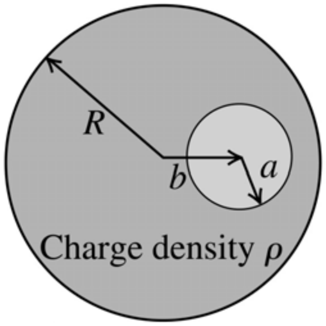
$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\,\frac{q}{d^2}\,\hat{d} = \frac{1}{4\pi\epsilon_0}\,\frac{q}{d^3}\,\vec{d}$$

$$q = \frac{4\pi b^3\rho}{3} \longrightarrow \vec{E}_2 = \frac{b^3}{3\epsilon_0 d^3}\,\rho\vec{d}$$

On the other hand, inside the sphere of radius a:

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\,\frac{q}{b^3}\,\vec{d} = \frac{\rho}{3\epsilon_0}\vec{d}$$

Recap of where we are:



- Want field anywhere for r<R
- Trick: add fields from
  1. solid sphere of radius R, q-density +ρ
  2. solid sphere of radius a, q-density -ρ

$$\vec{E}_1(\vec{r}) = \frac{\rho}{3\epsilon_0}\,\vec{r}$$

$$\vec{E}_2(\vec{r}) = \frac{\rho}{3\epsilon_0}\vec{d} \quad \text{or} \quad \vec{E}_2(\vec{r}) = \frac{b^3\rho}{3\epsilon_0}\,\frac{\vec{d}}{d^3}$$

inside the hole

outside the hole

$$\vec{d} = \vec{b} - \vec{r}$$

Now it is simply a matter to adding the two fields:

Inside the hole:

$$\vec{E} = \frac{\rho}{3\epsilon_0}\,\vec{r} \quad + \quad \frac{\rho}{3\epsilon_0}\,(\vec{b} - \vec{r})$$

$$\boxed{\vec{E} = \frac{\rho}{3\epsilon_0}\,\vec{b}}$$

Uniform!

Outside the hole:

$$\boxed{\vec{E} = \frac{\rho}{3\epsilon_0}\,[\vec{r} + \frac{b^3}{|\vec{r} - \vec{b}|^3}(\vec{b} - \vec{r})]}$$

Charges in a conductor

- In a conductor some of the electrons in the material are essentially free to move under the influence of electric fields.
- A conductor can have net negative charge, or net positive charge, or it can be neutral.
- Net negative charge if it acquired extra electrons from somewhere
  - e.g., another conductor
- Net positive charge if it gave away some of its electrons

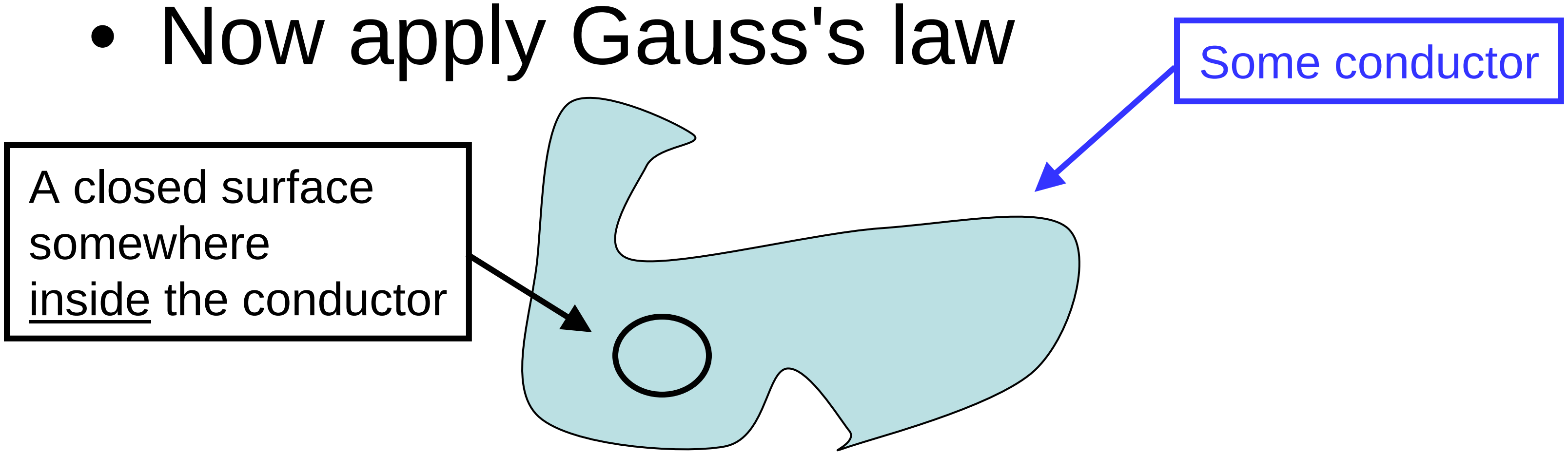
Charges in a conductor (cont.)

- We are concerned with electrostatic situations
  - electrostatic: the charges are not moving
- We said that in conductors some electrons are free to move under the influence of electric field

- In electrostatic situations the electric field must be zero everywhere inside the conductor
  - otherwise the free charges would be moving and the configuration would not be electrostatic anymore!

Charges in a conductor (cont.)

- Now apply Gauss's law



$$\Phi_E = \oint \vec{E} d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

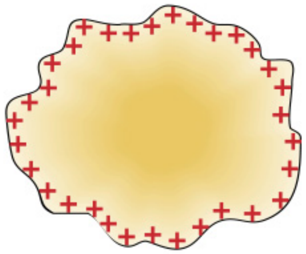
- The electric field is zero inside conductor
  - → Φ_E is zero for any closed surface (inside conductor)
  - → Q_{enclosed} by any surface is zero
- Make the volume enclosed by the surface infinitesimally small
  - → No net charge anywhere inside a conductor!!!

Charges in a conductor (cont.)

- We just showed that there can be no net charge anywhere inside a conductor
- Yet we know that we can charge-up a conductor
- So where does the charge go?

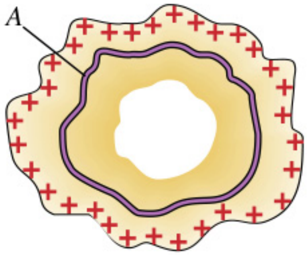
The excess charge on a conductor in an electrostatic situation is always on the surface

Cavities in a conductor



Solid charged conductor:
all the excess charge (+Q) is on the surface

Now suppose we have a cavity inside:

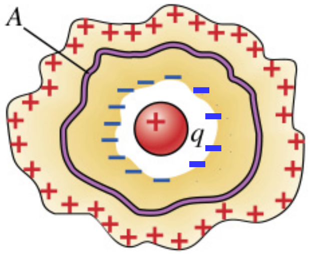


The flux through any gaussian surface *A* enclosing the cavity is zero because the field is zero in the conductor

→ the enclosed charge is zero

→ no charge on the surface of the cavity

Now suppose we place a charge inside the cavity:



The flux through the surface *A* is still zero.

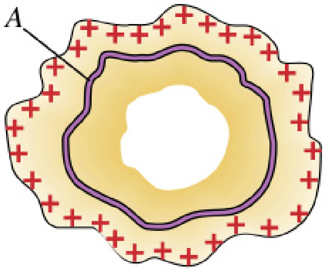
→ the total charge enclosed is zero

→ charge -q on the surface of the cavity

(we say that charge has been induced on the surface)

→ charge Q+q on the outer surface

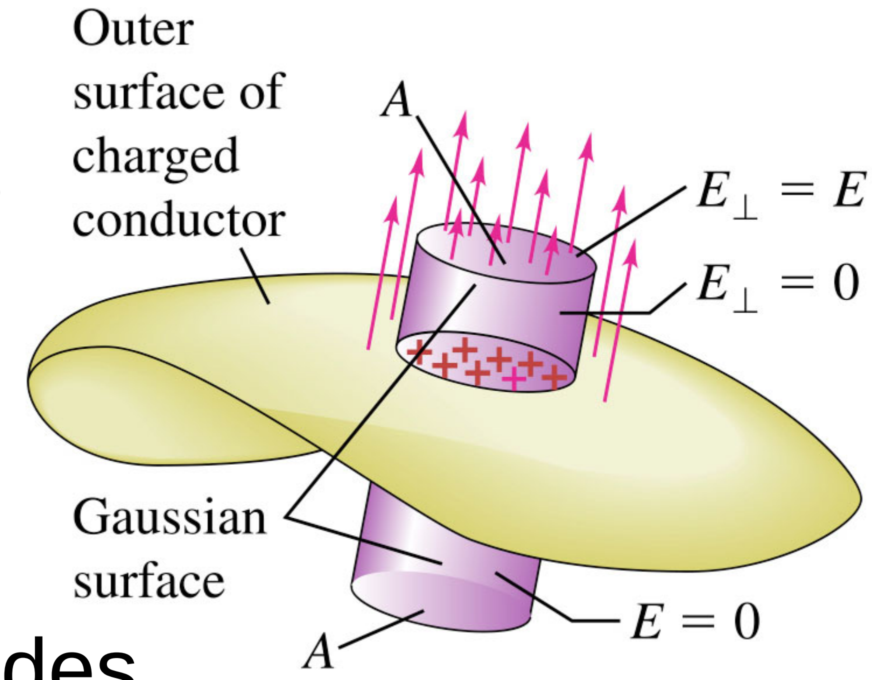
Look at the cavity again



- Inside the cavity there can be no electric field.
- If we want to shield a region of space from external electric fields, we can surround it by a conductor
- This is called a Faraday cage.

Field on conductor surface

$$\Phi_E = \oint \vec{E} d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$



- No flux through the sides
- No flux through the surface inside conductor
- Total flux = flux through top surface = AE

$$\Phi_E = AE = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

Irregularly shaped conductor

- e.g., fairly flat at one end and relatively pointed at the other.
- Excess of charge move to the surface.
- Forces between charges on the flat surface tend to be parallel to the surface.
- Charges move apart until repulsion from other charges creates equilibrium.
- At sharp ends, forces are predominantly directed away from surface.
- Less of tendency for charges located at sharp edges to move away from one another.
- Large σ and therefore large fields (and forces) near sharp edges.

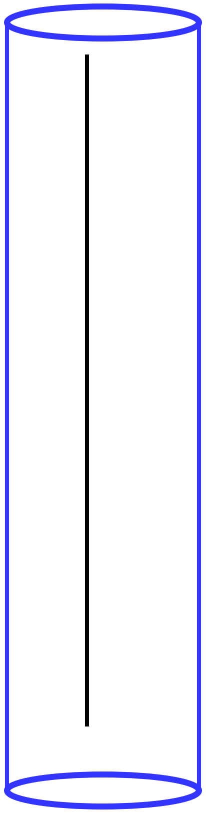
This is the principle behind the lightning rod.

Example Problem

Long straight wire surrounded by hollow metal cylinder. Axis of the wire coincides with axis of cylinder. Wire has charge-unit-length λ. Cylinder has charge per unit length 2λ.

(a) Find charge-per-unit-length on inner and outer surfaces of cylinder.

(b) The electric field outside the cylinder a distance *r* from the axis.



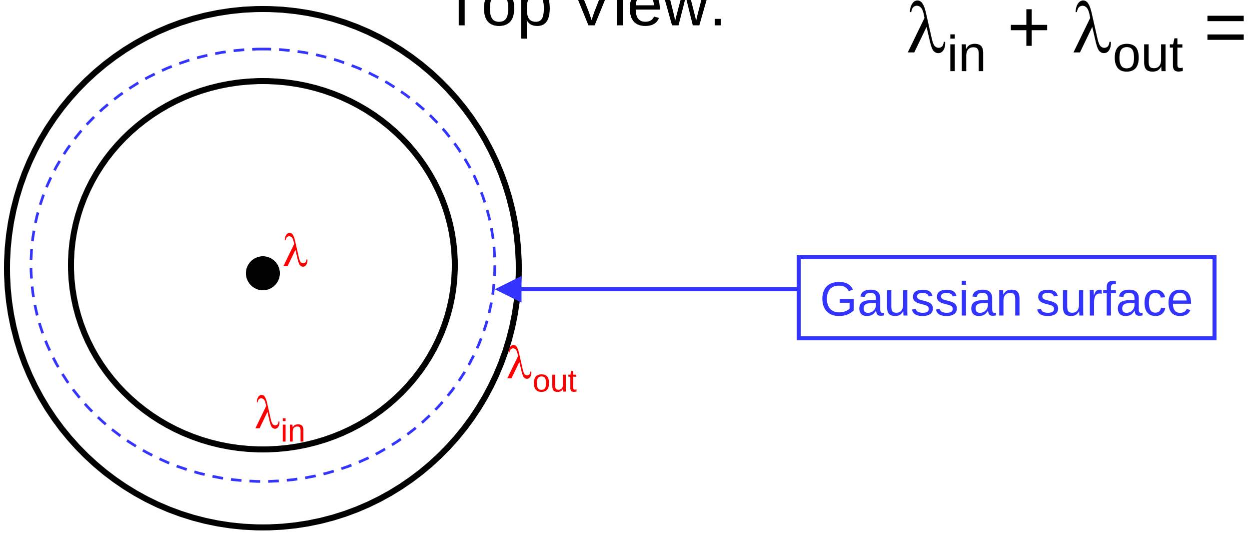
λ_{in} = charge-unit-length on inside of cylinder

λ_{out} = charge-unit-length on outside of cylinder

$$\lambda_{in} + \lambda_{out} = 2\lambda$$

Top View:

$$\lambda_{in} + \lambda_{out} = 2\lambda$$



Choose a gaussian cylindrical surface with same axis but with radius in between the inner and outer radius.

Flux through this surface is zero. Because the electric field in the conductor is zero.

→ By Gauss's law, total charge enclosed is zero

→ $\lambda + \lambda_{in} = 0$ so $\lambda_{in} = -\lambda$.

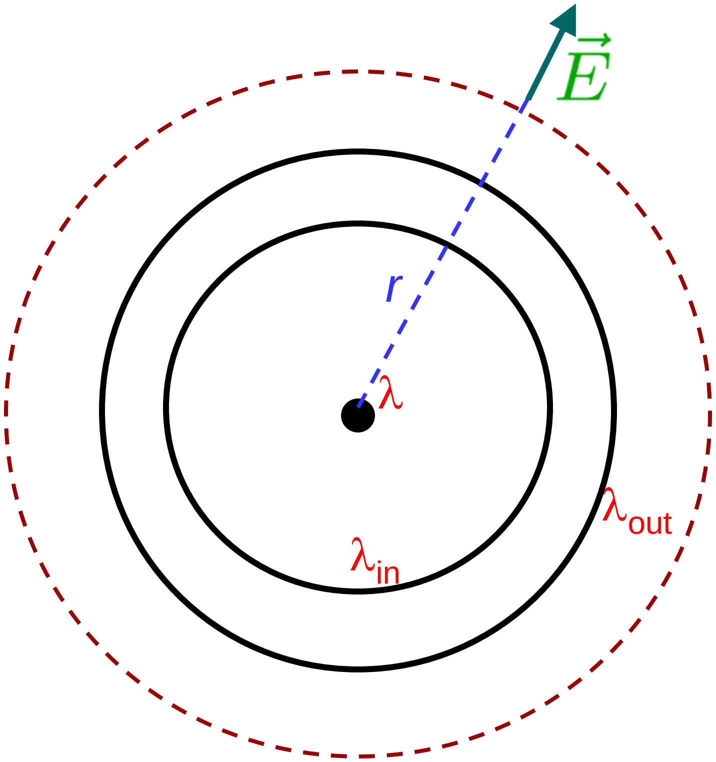
But $\lambda_{in} + \lambda_{out} = 2\lambda$

→ $\lambda_{out} = 3\lambda$

Now want the field a distance *r* from the axis (outside cylinder)

$\lambda_{in} = -\lambda$

$\lambda_{out} = 3\lambda$



By symmetry, electric field can only point radially.

Draw Gaussian cylindrical surface of radius *r*

$\Phi_E = E(r)\ C\ d$

where C = circumference cylinder

d = length of cylinder

But $C = 2\pi\ r$

→ $\Phi_E = 2\ \pi\ E(r)\ d\ r$

Gauss's Law: $\Phi_E = Q_{enclosed}/\varepsilon_0$

$\Phi_E = (\lambda + \lambda_{in} + \lambda_{out})d/\varepsilon_0$

→ $\Phi_E = 3\lambda\ d/\varepsilon_0$

$$\boxed{E(r) = \frac{3}{2\pi\epsilon_0}\frac{\lambda}{r}}$$

Directed outward if $\lambda>0$

Directed inward if $\lambda<0$

Example Problem

The electric field on the surface of an irregularly shaped conductor varies between 56 kN/C and 28 kN/C. Calculate the local surface charge density at the point on the surface where the radius of curvature is maximum or minimum

$$E = \frac{\sigma}{\epsilon_0}$$

Maximum electric field when charge density is highest. This happens when the surface has sharp edges, i.e., when the radius of curvature is minimum.

$$\sigma_{\text{max}} = \epsilon_0 E_{\text{max}}$$

$$\sigma_{\text{max}} = 8.85 \times 10^{-12} \cdot 56,000 \frac{C}{m^2} = 5.0 \times 10^{-7} \frac{C}{m^2}$$

Minimum electric field when charge density is lowest. This happens when the surface is flattest, i.e., when the radius of curvature is maximum.

$$\sigma_{\text{max}} = \epsilon_0 E_{\text{max}}$$

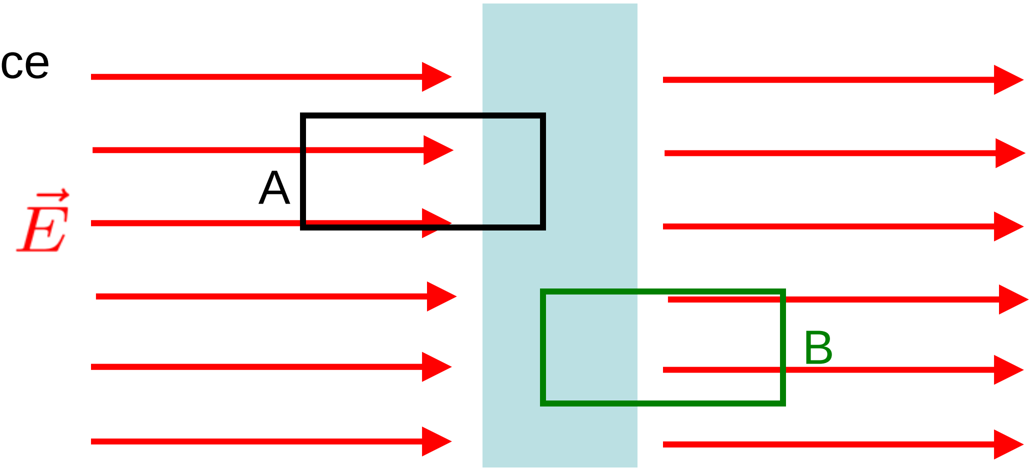
$$\sigma_{\text{max}} = 8.85 \times 10^{-12} \cdot 28,000 \frac{C}{m^2} = 2.5 \times 10^{-7} \frac{C}{m^2}$$

Example Problem

A square plate of Cu with 50 cm sides has no net charge. It is placed in a region of uniform electric field 80 kN/C directed perpendicular to the plate. Find

(a) The charge density on each face

(b) The total charge on each face



1. Make a drawing
2. Pick gaussian surfaces
   - behind (box A)
   - in front (box B)
3. Get Φ_A and Φ_B

There is no electric field inside the conductor and the electric field is parallel to the "long" sides of the boxes.

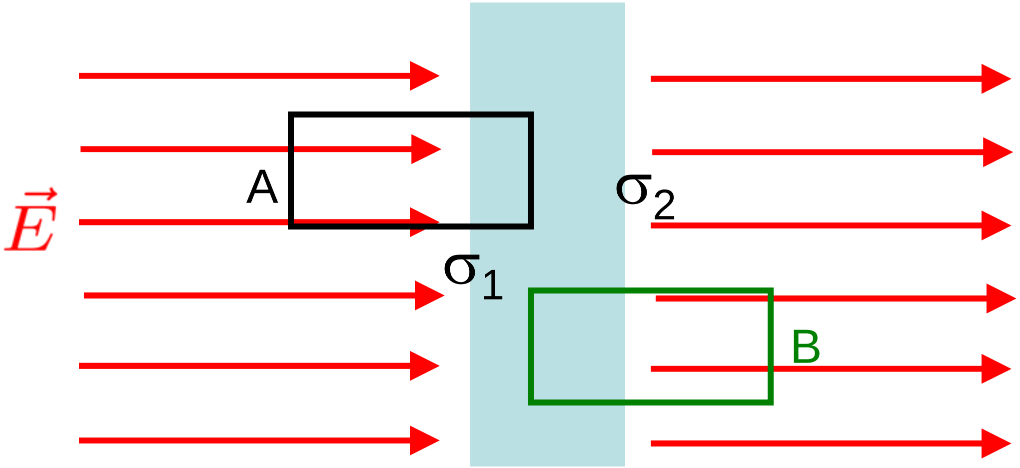
→ only contribution to Φ_A and Φ_B are from the vertical surfaces outside the conductor

Let S = area of vertical surfaces if gaussian boxes

Φ_A = - ES (field goes into the box)

Φ_B = + ES (field goes out of the box)

$\boxed{\Phi_A = -ES \text{ and } \Phi_B = +ES}$



The charge enclosed in box A is $S\sigma_1$
The charge enclosed in box B is $S\sigma_2$

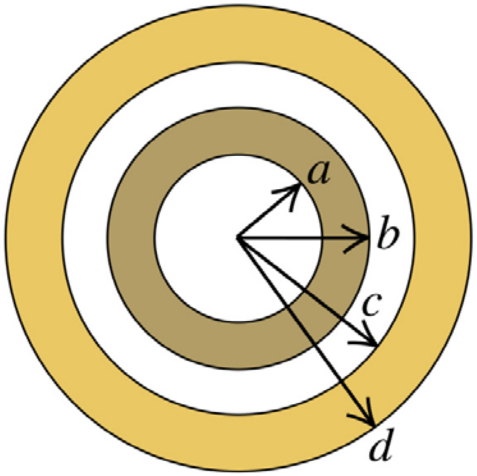
Then, by Gauss's Law:
$\sigma_1 = -\varepsilon_0 E = -8.85\ 10^{-12} \times 80{,}000\ C/m^2 = -7.1\ 10^{-7}\ C/m^2$
$\sigma_2 = -\sigma_1 = +7.1\ 10^{-7}\ C/m^2$

Each surface has area $(50\ cm)^2 = 0.25\ m^2$
→ each surface has charge $0.25 \times 7.1\ 10^{-7}\ C = 1.8\ 10^{-7}\ C$

Example Problem (22.39)

Concentric conducting spherical shells

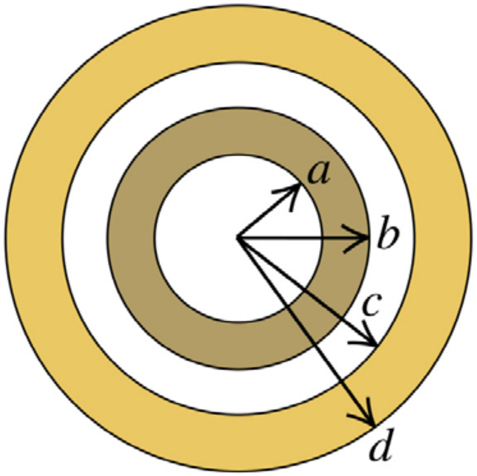
Inner shell: charge $+2q$
Outer shell: charge $+4q$
Calculate electric field for
(a) $r<a$
(b) $a<r<b$
(c) $b<r<c$
(d) $c<r<d$
(e) $r>d$



Some of these answers are trivial.

- In cases (b) and (d) the field is zero (inside conductor)
- In case (a) the field is also zero (Faraday cage!)

Inner shell: charge +2*q*
Outer shell: charge +4*q*
Want field for $b<r<c$ and $r>d$

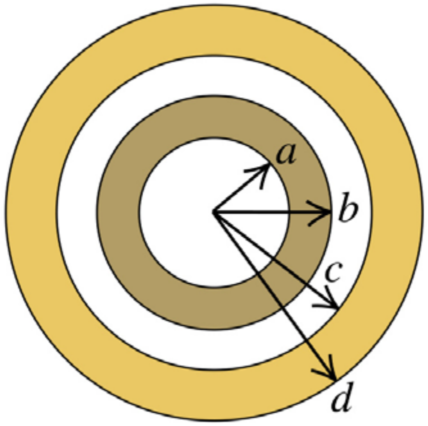


First, note that by symmetry the field can only be radial.
Then, construct spherical gaussian surface of radius *r*
Area of the surface = $4\pi r^2$
Flux through the surface =$4\pi\, r^2 E(r)$
Charge enclosed:

- If $b<r<c$, $Q_{\text{encloded}} = 2q$
- If $r>d$, $Q_{\text{enclosed}} = 6q$

$$\boxed{\begin{aligned} E(r) &= \frac{1}{2\pi\epsilon_0}\,\frac{q}{r^2} \qquad (b<r<c) \\ E(r) &= \frac{3}{2\pi\epsilon_0}\,\frac{q}{r^2} \qquad (r>d) \end{aligned}}$$

Small shell: charge +2*q*
Large shell: charge +4*q*



Next question:
What are the charges on the four surfaces?

Inner surface of small shell (r=a).

- No electric field → no charge on this surface

Outer surface of small shell (r=b)

- Total charge on small shell = +2q.
- There is no charge on the other surface of this conductor
- All the charge of the shell (+2q) must be on this surface!

Inner surface of large shell (r=c)

- Gaussian spherical surface, c<r<d.
- No field (in conductor!) → no flux → enclosed charge = 0
- Charge on this surface + charge on small shell (=+2q) must add to 0
- Charge on this surface = -2q

Outer surface of large shell (r=d)

- Total charge on large shell = + 4q. Charge on other surface = -2q
- Charge on this surface = + 6q