Fall 2004 Physics 3
Tu-Th Section

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Lecture 8: 19 Oct. 2004

Web page:
http://hep.ucsb.edu/people/claudio/ph3-04/

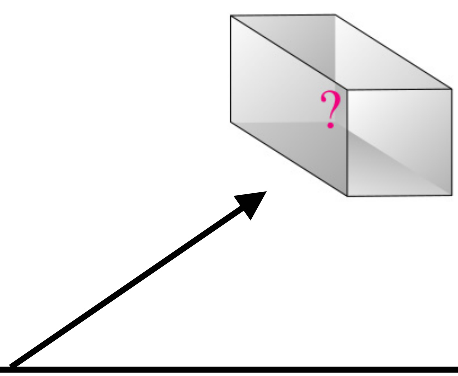
Midterm Reminder

- The midterm is next Tuesday (Oct 26).
- Open book and open notes
- Bring
  - Picture ID
  - Blue Book
  - Calculator
- There will be a review session, Sunday Oct 24, 7 pm → ?, Broida 1610.
- The midterm will cover chapters 15-16-21
  - This week is chapter 22
- I have posted exams from previous years on the physics 3 website
  - Warning: these are exams prepared by different instructors, but they should still be very useful
- Exam will have a mix of multiple choice and non-multiple choice questions

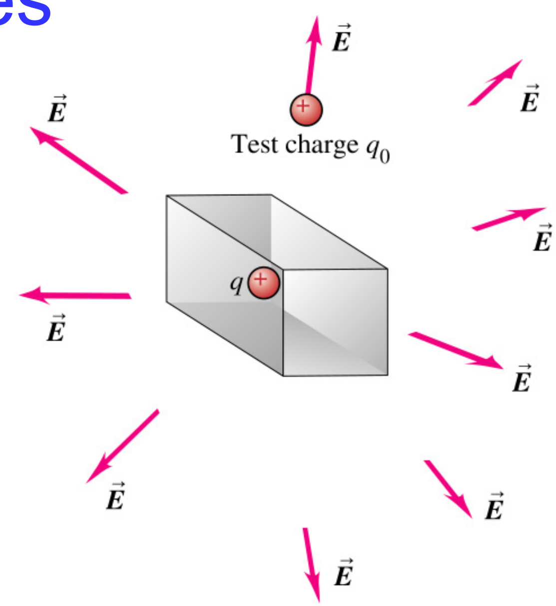
Today: Gauss's Law

- A different way of thinking of the relationship between charges and the electric field that they cause
- Reminder: up to now, when we wanted electric field due to a bunch of charges:
  1. Calculate the electric field due to each charge
  2. Add (vectorially) all the electric fields
- Gauss's law takes a different approach which emphasizes the symmetry of the problem
  - Often makes things a lot simpler!

- Coulomb's law:
  - ➢ Start with the charges tells us something about the electric field
- Gauss's law:
  - ➢ Start with the electric field, tells us something about the charges



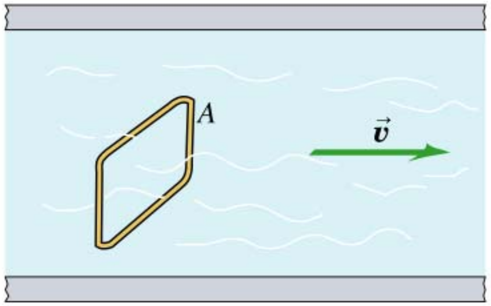
Imaginary box containing unknown charge



By measuring E outside the box, get information on charge inside

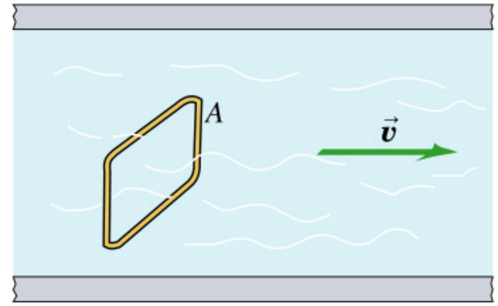
Flux of electric field

- We will see that Gauss's law is expressed in terms of the electric field flux. Let's try to define it!
- Analogy with the flux of a fluid:
- In this picture, water "flows" through the wire rectangle



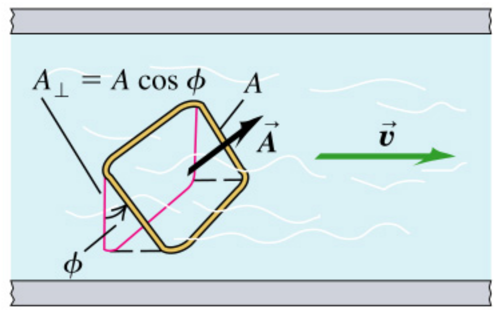
- What would be a sensible mathematical definition of the flux of H_2O through area A?
  - ➢ Should be proportional to A
    - More area, more "flux"
  - ➢ Should be proportional to H_2O velocity
    - More velocity, more H_2O flows through the area

- Define flux: $\Phi_{H_2O} = vA$
- This is actually = dV/dt



where V = volume of fluid going through area:

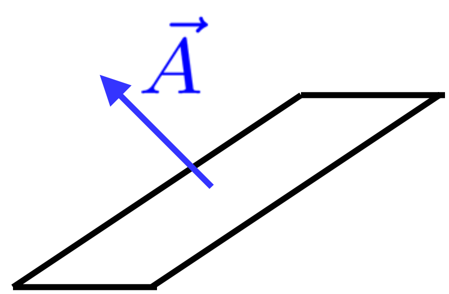
- In time Δt, a volume of fluid $\Delta V = A\,(v\Delta t)$ passes through → $\Delta V/\Delta t = vA$



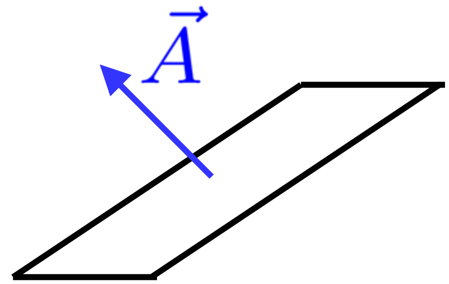
- What if A is "tilted"?
- Projection of area perp. to flow is $A\cos\phi$
- Therefore:

$$\Phi_{H_2O} = \frac{dV}{dt} = vA\cos\phi$$

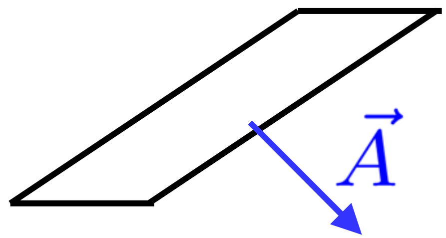
- Introduce a new concept: vector area
- We know about "area". We now also want to specify the orientation of the area in space.
- This can be done by defining vector area as a vector with
  - Magnitude = area of the plane
  - Direction perpendicular to the plane



- Careful about ambiguity!!!
- Why did I choose it as

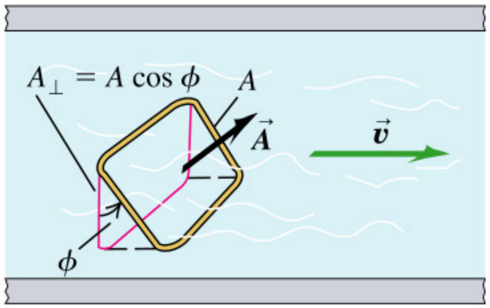


- Instead of



- The choice is arbitrary
  - But I need to specify it!

Back to H_2O flux



$$\Phi_{H_2O} = \frac{dV}{dt} = vA\cos\phi$$

Rewrite this in terms of vector area:

$$\boxed{\Phi_{H_2O} = \frac{dV}{dt} = \vec{v}\cdot\vec{A}}$$

Remember:

- Flux always defined with respect to some surface
- Need to specify "direction" of surface

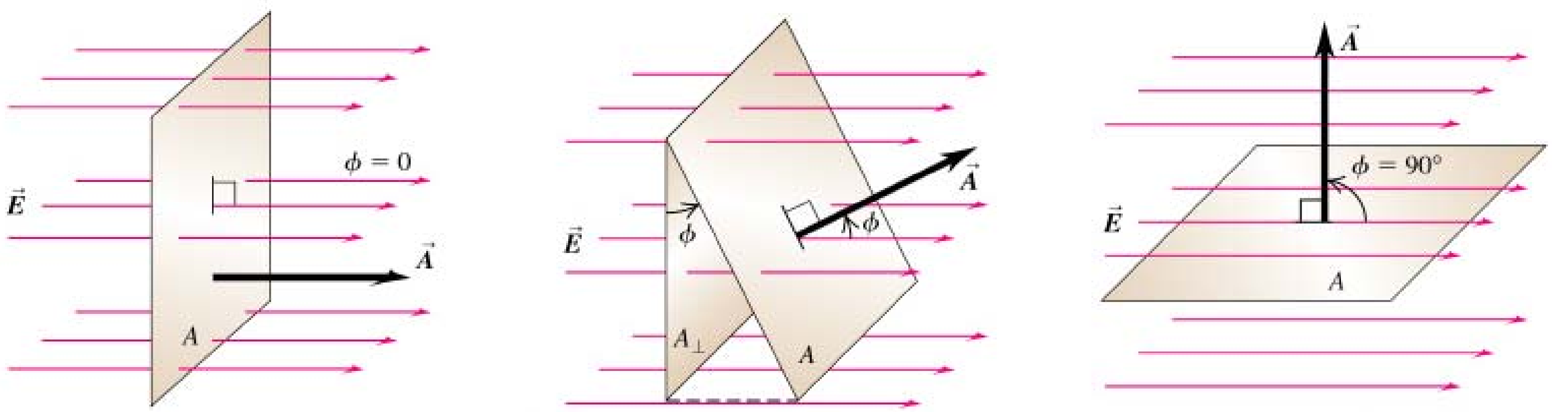
Electric field flux

Define electric field flux in analogy to H_2O flux

velocity of H_2O ←→ electric field

$$\Phi_{H_2O} = \vec{v} \cdot \vec{A}$$

$$\Phi_E = \vec{E} \cdot \vec{A}$$



(a) Surface face-on to electric field
$\vec{E}$ and $\vec{A}$ parallel
angle between $\vec{E}$ and $\vec{A}$ is $\phi = 0$
flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$

(b) Surface tilted from face-on orientation by an angle ϕ
angle between $\vec{E}$ and $\vec{A}$ is ϕ
flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$

(c) Surface edge-on to electric field
$\vec{E}$ and $\vec{A}$ perpendicular angle between $\vec{E}$ and $\vec{A}$ is $\phi = 90°$
flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos 90° = 0$

$$\Phi_E = \vec{E} \cdot \vec{A}$$

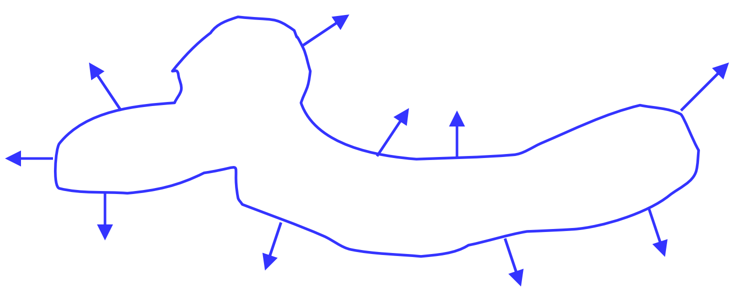
- If the electric field is not constant over the area, or if the area is not a simple plane so that the vector A points in different directions at different points, then:
  - Calculate $d\Phi_E$ for an infinitesimally small area
  - Add (i.e. integrate) the contributions over all the small areas

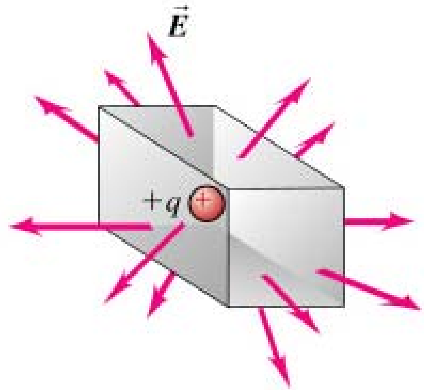
$$\Phi_E = \int E \cos\phi \, dA = \int E_\perp \, dA = \int \vec{E} \cdot d\vec{A}$$

This is called a surface integral

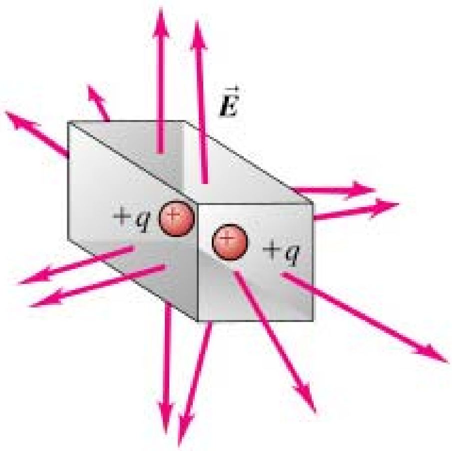
Towards Gauss's Law

- Gauss's Law is a relationship between the total flux through a closed surface and the total charge enclosed by the surface
- Let's try to justify it first. We will show (rigorously) that it is correct later
- The flux through a closed surface is the sum of the fluxes through all of the pieces of area that make up the surface.
- Convention: we take the vector areas at any point to always point outwards

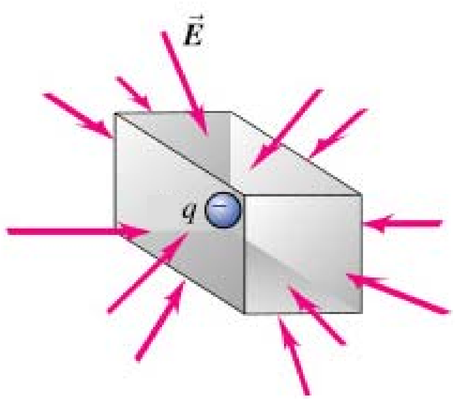




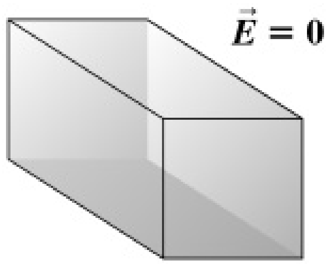
Surround positive charge with imaginary box
Electric field flux points outward
Φ_E positive since area vectors point outward



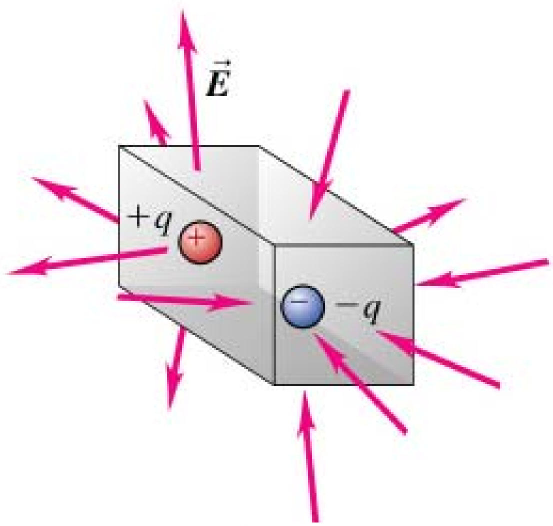
Place a second +ve charge in the box.
Φ_E increases



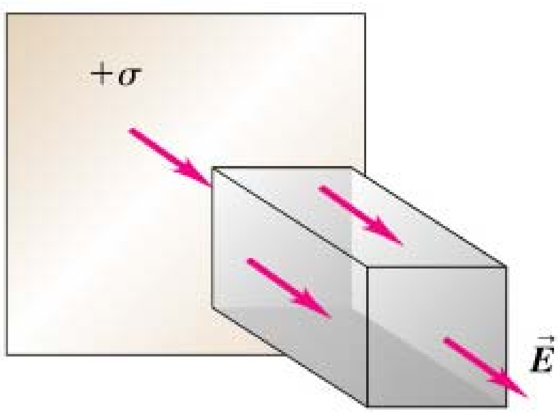
Negative charge in the box.
Electric field direction reversed w.r.t. 1st example.
Φ_E is negative
(but in magnitude equal to the 1st example)



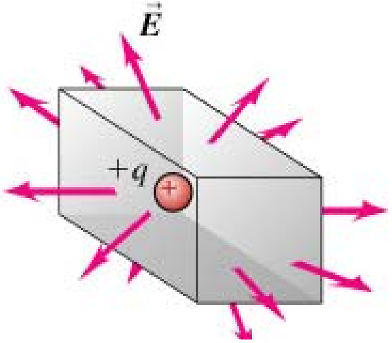
No charges in the box
No electric field
$\Phi_E = 0$



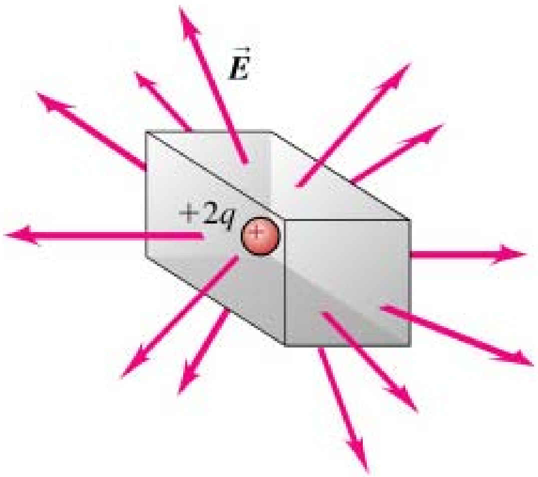
One positive and one negative charge in the box
The electric fields, and therefore the fluxes, due to the two charges tend to cancel each other out



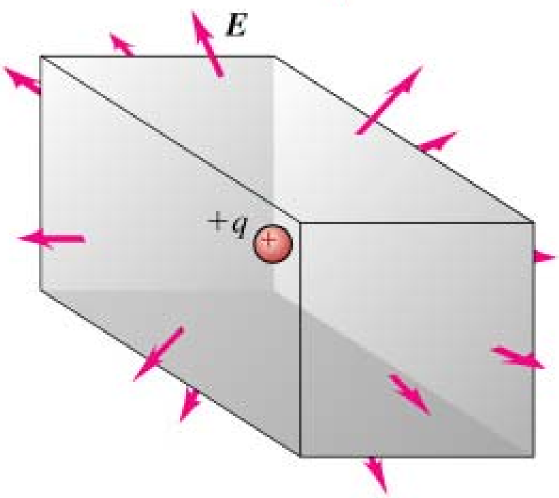
Empty box in constant E-field
The flux through the long sides is zero
(because E is parallel to surface)
The flux through the far side is –ve
(because E points into the box)
The flux through near side is +ve
(because E points out of the box)
→ total $\Phi_E = 0$



We had this already:
Electric field flux points outward
Φ_E positive since area vectors point outward



Make the charge twice as big.
Same box.
The electric field doubles everywhere.
Φ_E also doubles



Keep the charge the same.
Double the size of the box.
This doubles the distance from the charge to any point on the box.
Since $E \sim 1/r^2$, the electric field on the surface of the box is down by a factor of ¼
But the surface of the box increases by a factor of 4 → Φ_E stay the same.

Gauss's Law

These examples suggest the following relationships between Φ_E over a closed surface and the enclosed charge (Q)

1. Φ_E is positive if Q is positive, negative if Q is negative
2. Φ_E is zero if Q is zero
3. Φ_E is proportional to Q
4. Φ_E is independent of the shape of the closed surface (the box)

Mathematical formulation of Gauss's law

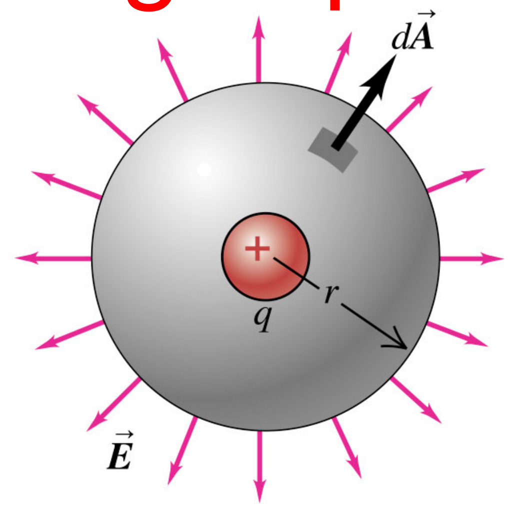
Plan of attack:

1. Formulate it for a single point charge and spherical surface
2. Show that it works also for a single point charge and <u>any</u> closed surface
3. Show that it works for <u>any</u> collection of charges for <u>any</u> surface

Each result will build on the previous one!

1. Point charge, flux through sphere

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

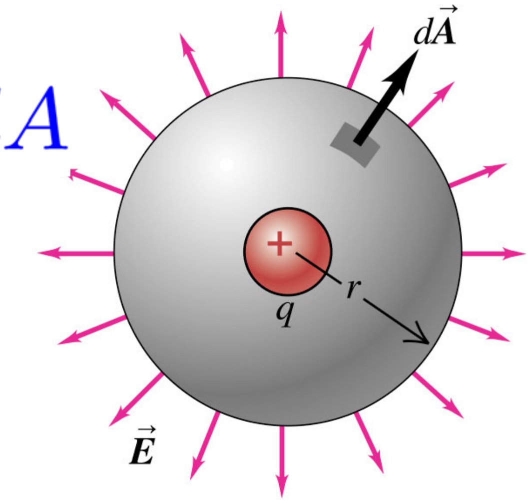


- At any point, the vector *dA* points <u>outward</u> and <u>perpendicular</u> to <u>surface</u> of imaginary sphere
  - *dA* points radially outward
- Electric field also points radially outward (q>0)
- Electric field magnitude is constant over the surface

$$\Phi_E = \int \vec{E} \cdot d\vec{A} = \int E dA = E \int dA$$

1. Point charge, flux through sphere

$$\Phi_E = \int \vec{E} \cdot d\vec{A} = \int E dA = E \int dA$$



$$E = k\,\frac{q}{r^2} = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}$$

Surface of sphere: $\int dA = 4\pi r^2$

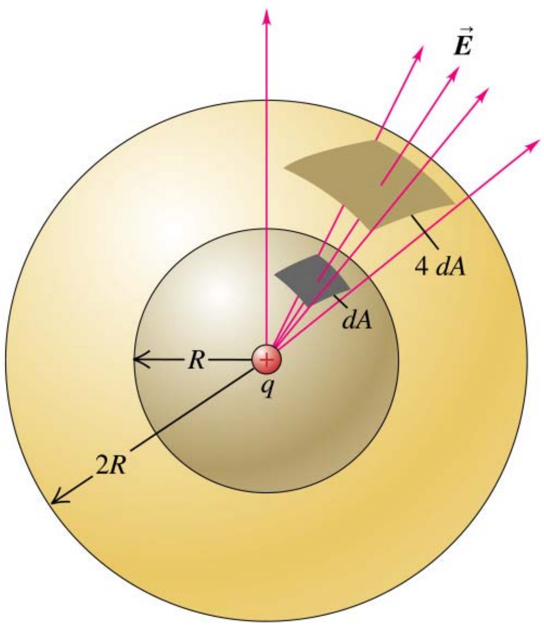
$$\boxed{\longrightarrow \Phi_E = \oint \vec{E} d\vec{A} = \frac{q}{\epsilon_0}}$$

This is independent of the size of the sphere!

Note $\oint$ means integral over a closed surface

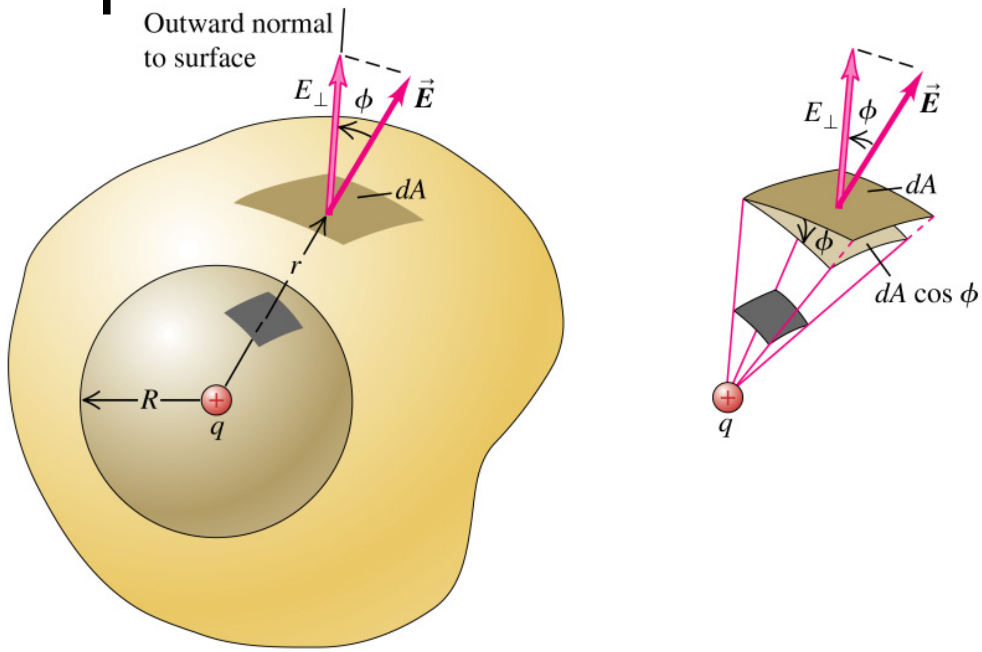
2. Result hods for <u>any</u> surface

- Start with two spheres radii, R and 2R
- Have already shown that flux is the same
- What is actually going on?



- Projection of area *dA* onto bigger sphere
- New area 4*dA*
- Field down by a factor 4 in going from R→2R ($1/r^2$ dependence)
- Flux (field*area) is unchanged

- Now go from a sphere of radius R to an arbitrary shape



- The projected area is always larger than what it would have been if the shape was also a sphere
  - $dA = dA \cos\phi$

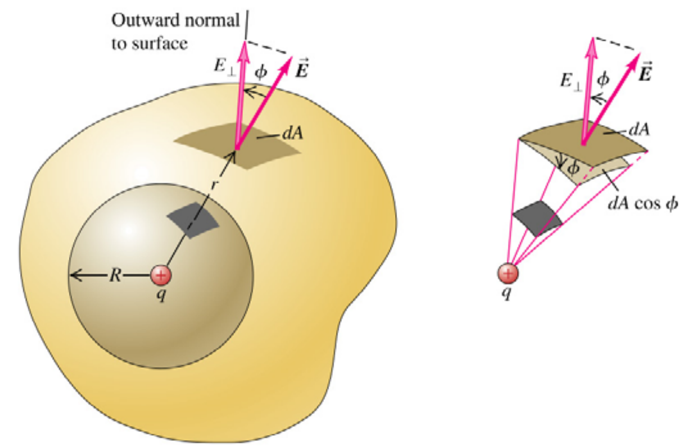
Area on sphere

Area on arbitrary surface

$dA = dA\cos\phi$

Area on sphere

Area on arbitrary surface



Flux through element if the shape was a sphere:

$$d\phi_E = E\,\boxed{(dA\cos\phi)}$$

Area of element if shape was sphere

SAME

Flux through element of arbitrary shape

$$d\Phi_E = \vec{E}\cdot d\vec{A} = E\,dA\cos\phi$$

→ Shape of surface does not matter !!!!!

3 Generalize to ≥ 1 enclosed charges

- For one charge $\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$ regardless of the shape of the surface
- For more than one charge, the total flux through the surface is the sum of the fluxes due to all the charges enclosed by the surface
  - $\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \ldots..$
  - $\Phi_E = q_1/\varepsilon_0 + q_2/\varepsilon_0 + q_3/\varepsilon_0 + \ldots..$

Then, Gauss's Law is:

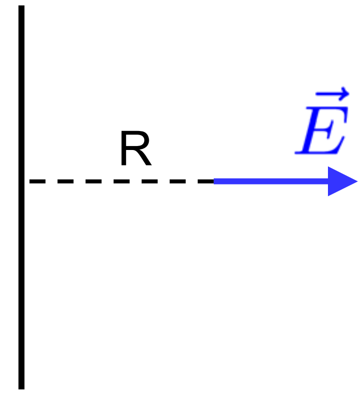
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{total}}}{\epsilon_0}$$

Q_{total} = total charge enclosed by the surface

Example 1

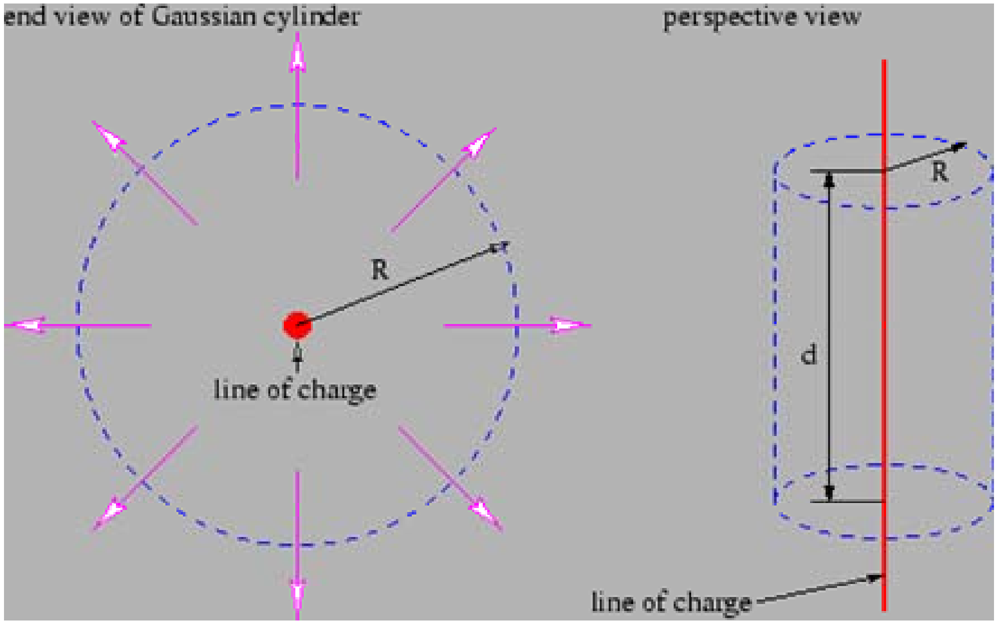
Infinitely long line of charge,
λ = charge-per-unit-length:

$$E = k\frac{2\lambda}{R} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{R}$$



We did this last week using Coulomb's law. Now try using Gauss' law.

1. Choose appropriate surface
   - Base your choice on <u>symmetry</u> considerations
2. The system has <u>cylindrical</u> symmetry
   - Choose a cylindrical surface
3. Also, because of cylindrical symmetry the electric field can only be radially outward/inward from the wire



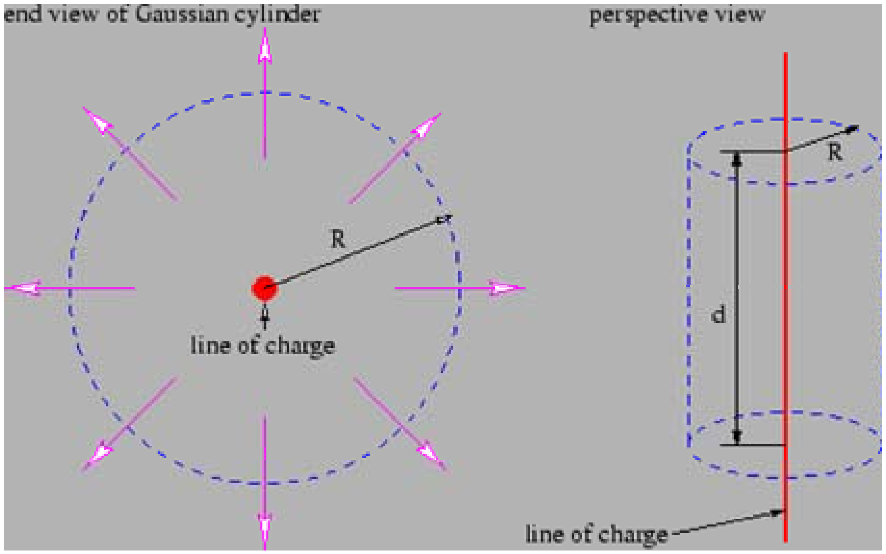
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\mathrm{total}}}{\epsilon_0} = \frac{\lambda d}{\epsilon_0}$$

Break the flux into a sum over the pieces of the surface

$\Phi_E = \Phi_{top} + \Phi_{side} + \Phi_{bottom}$

But $\Phi_{bottom} = \Phi_{top} = 0$ because E-vector is in plane of bottom and top surface

$$\boxed{\Phi_E = \Phi_{side}}$$



$$\boxed{\Phi_E = \Phi_{side}}$$

$$\boxed{\Phi_E = \frac{\lambda d}{\epsilon_0}}$$

$E_{side} = E(R)$

$\Phi_{side} = E_{side}$ Area

$\rightarrow \Phi_{side} = \Phi_E = 2\pi$ R d E(R)

Area = $\underbrace{2\pi R}_{\text{circumference}}\ \underbrace{d}_{\text{height of cylinder}}$

$$\frac{\lambda d}{\epsilon_0} = 2\pi \cdot R \cdot d \cdot E(R)$$

$$\boxed{E(R) = \frac{1}{2\pi\epsilon_0}\,\frac{\lambda}{R}}$$

Note that the height of the cylinder drops out, as it should!

Example 2

What is the field of a uniformly charged sphere?

Note: we'll show later that the the charge on a conductor is all on the surface → this would be a sphere made of some insulator.

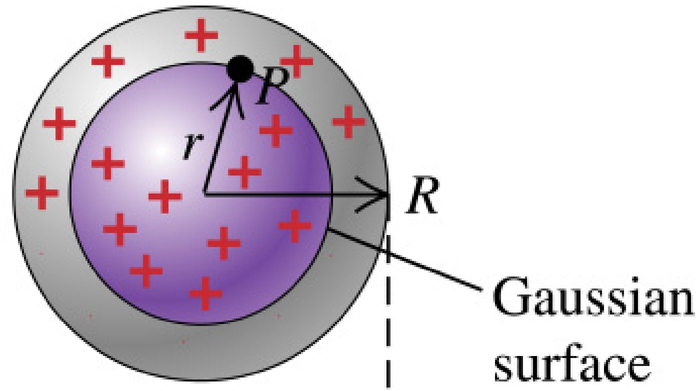
- Choose appropriate surface
  - Base your choice on <u>symmetry</u> considerations
- The system has <u>spherical</u> symmetry
  - Choose a spherical surface
- Also, because of spherical symmetry the electric field can only be radially outward/inward from the center of sphere

Let: R = radius of the sphere
Q = charge of the sphere

$$\Phi_E = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Must distinguish between two cases:

1. r < R (field <u>inside</u> the sphere)
2. r > R (field <u>outside</u> the sphere)



In both cases

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E(r)$$

If r>R, then enclosed charge is Q

If r<R, then enclosed charge Q_{enclosed} is

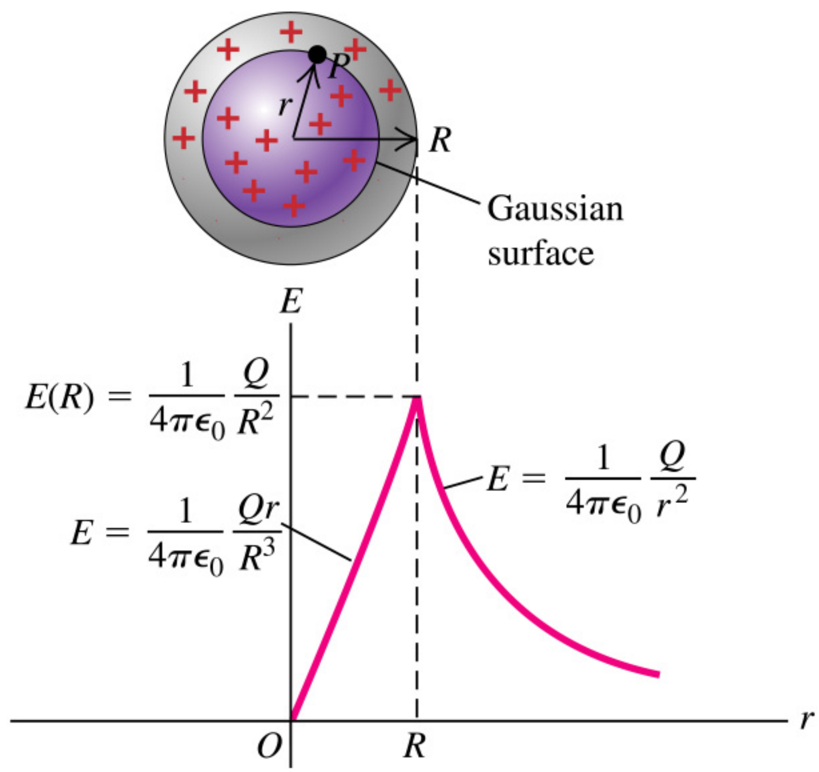
$$Q \, \frac{\text{Volume of sphere of radius r}}{\text{Volume of sphere of radius R}} = \frac{4\,\pi\, r^3}{4\,\pi\, R^3}\, Q = \frac{r^3}{R^3}\, Q$$

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E(r) = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Outside the sphere, $Q_{\text{enclosed}} = Q$:

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

Same as for point charge!!



Inside the sphere,
$Q_{\text{enclosed}} = Q\,(r/R)^3$:

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3}$$

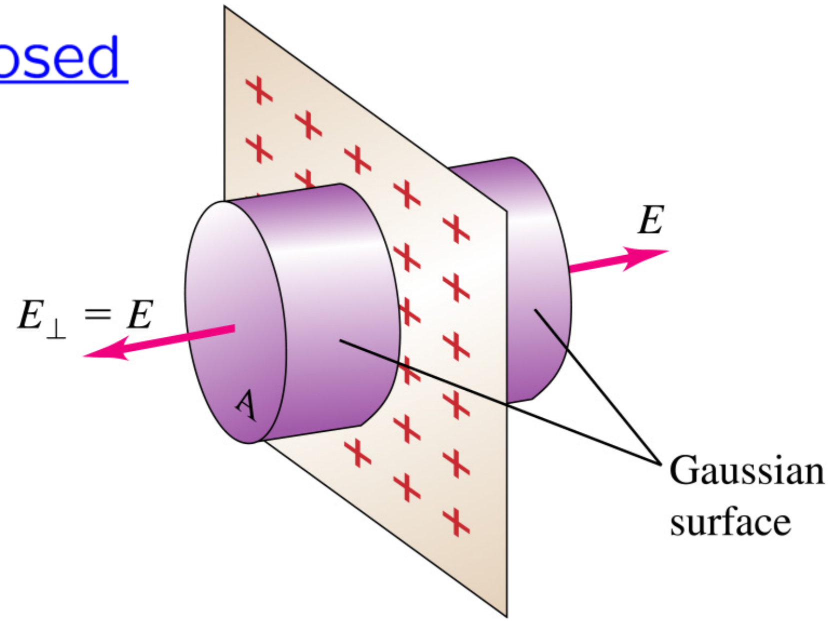
Example 3

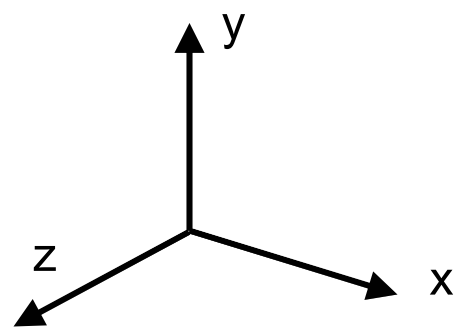
Field of a uniformly charged infinite sheet

(we did this last week also)

- Choose appropriate surface
  - Base your choice on <u>symmetry</u> considerations
  - Call the plane of the sheet the (xy) plane
- The system has <u>translational</u> symmetry in the (xy) plane
- Also, because of symmetry the electric field can only be in the z-direction

$$\Phi_E = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$





Note: I could have just as well picked a box shape instead of a cylinder. What is important (in order to simplify the calculation) is to pick the sides parallel and perpendicular to the (xy) plane!

$Q_{\text{enclosed}} = \sigma A$ (σ = charge-unit-area)

Φ through the sides is zero (field parallel to surface)

→ $\Phi = 2EA = Q_{\text{enclosed}}A$

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

independent of choice of A

Another example (Problem 22.52)

A Hydrogen atom has a proton of charge Q=1.6 10^{-19} C and an electron of charge –Q. Take the proton as a point charge at r=0. The motion of the electron causes its charge to be "smeared out" into a spherical distribution around the proton. The electron is equivalent to a charge per unit volume

$$\rho(r) = -\frac{Q}{\pi a_0^3} e^{-2r/a_0}$$

a_0 = 5.3 10^{-11} m is called the Bohr radius.

Find the electric field, magnitude and direction, as a function of r.

Solution to Problem 22.52

- Principle of superposition:
  - Find the field due to proton
  - Find the field due to electron
  - Add them up (vectorially!)
- Field due to proton is trivial (point charge)
  - $E = (1/4\pi\varepsilon_0)\,(Q/r^2)$
  - Direction: radially outward
- Field due to electron
  - Use Gauss's law.
  - Problem has spherical symmetry
    - → Choose a spherical surface of radius R

Gauss's Law: $$\Phi_E = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Consider a sphere of radius R.
What is the electron charge enclosed by the sphere?

$$Q_{\text{enclosed}} = \int \rho(r)\,dV$$

The integral is over the sphere.
To do it, consider spherical shell, radius *r*, thickness *dr*
Then $dV = 4\pi r^2\,dr$
(since $4\pi r^2$ is the area of the shell)

$$Q_{\text{enclosed}} = \int_{r=0}^{r=R} \rho(r) 4\pi r^2 dr$$

$$Q_{\text{enclosed}} = -\frac{4Q}{a_0^3}\int_{r=0}^{r=R} e^{-2r/a_0} r^2 dr$$

$$Q_{\text{enclosed}} = -\frac{4Q}{a_0^3}\int_{r=0}^{r=R} e^{-2r/a_0} r^2 dr$$

Use table of integrals:

$$\int x^2 e^{cx} dx = e^{cx}\left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right)$$

$$Q_{\text{enclosed}} =$$

$$-\frac{4Q}{a_0^3}\left[\frac{a_0^3}{4} - \frac{1}{2}e^{-2R/a_0}\left(a_0 R^2 + a_0^2 R + \frac{a_0^3}{2}\right)\right]$$

$$Q_{\text{enclosed}} = -Q\left[1 - 2e^{-2R/a_0}\left(\frac{R^2}{a_0^2} + \frac{R}{a_0} + \frac{1}{2}\right)\right]$$

Sanity checks:	R=0	$Q_{\text{enclosed}} = 0$	OK!
	$R \to \infty$	$Q_{\text{enclosed}} = -Q$	OK!

Gauss's Law: $\Phi_E = \frac{Q_{\text{enclosed}}}{\epsilon_0}$

And we now have Q_{enclosed} !

Next step: get the flux in terms of E-field

$\Phi_E = E(r)\, 4\pi\, R^2$

So, field due to electron:

$$E(R) = -\frac{Q}{4\pi\epsilon_0 R^2}[1 - 2e^{-2R/a_0}(\frac{R^2}{a_0^2} + \frac{R}{a_0} + \frac{1}{2})]$$

Directed radially inwards (-ve sign)

Field due to proton was radially outward and

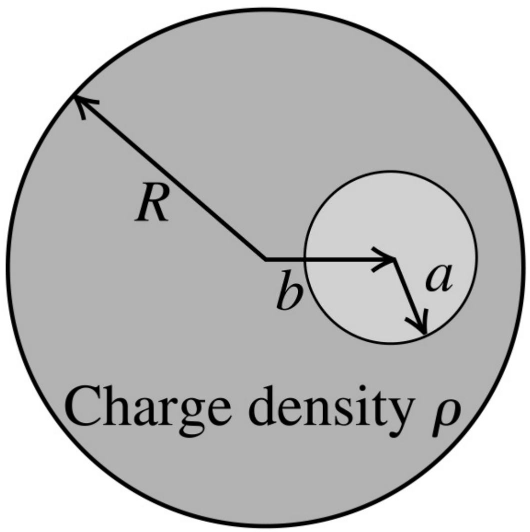
$$E(R) = \frac{Q}{4\pi\epsilon_0 R^2}$$

And adding the two we finally get:

$$\boxed{E(R) = \frac{Q}{4\pi\epsilon_0 R^2}\, 2e^{-2R/a_0}(\frac{R^2}{a_0^2} + \frac{R}{a_0} + \frac{1}{2})}$$

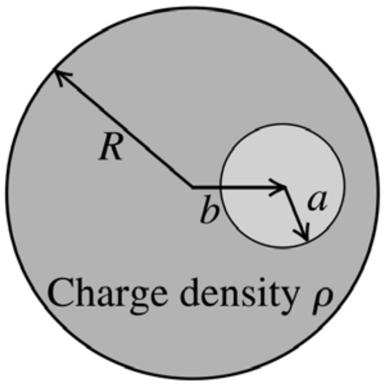
Another example (similar to Problem 22.61)

Insulating sphere of radius *R* charge density ρ.
The sphere has a hole of radius *b*.
Find the E field in the insulator and in the hole.



Trick: use principle of superposition:

1. solid sphere radius *R*, charge density ρ
2. solid sphere, radius *a*, charge density $-\rho$



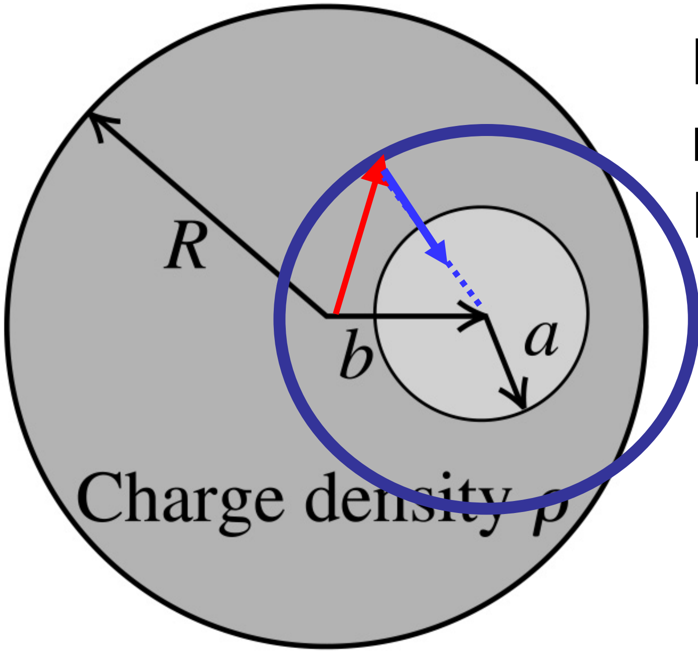
Trick: use principle of superposition:
1. solid sphere radius *R*, charge density ρ
2. solid sphere, radius *a*, charge density -ρ

Use result of "example 2" for field of 1:

$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{Q}{R^3}\,\vec{r}$$

Here I wrote it as a vector equation. The r-vector points from the center of the big sphere to the point at which we want E. Recast this using $\rho = (Q/V)$ and $V = 4\pi R^3/3$

$$\vec{E}_1(\vec{r}) = \frac{\rho}{3\epsilon_0}\,\vec{r}$$



Now the field due to the fictitious negative charge density in the hole. First, look outside the hole.

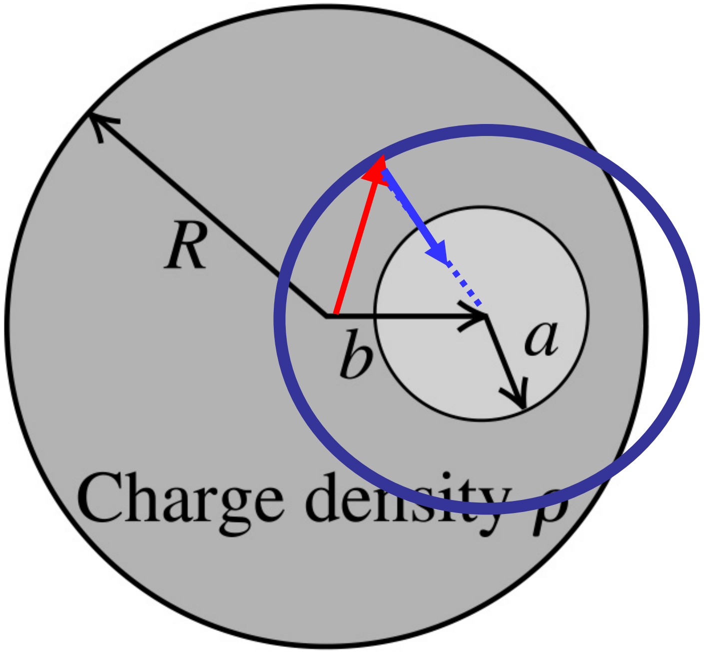
Draw imaginary (gaussian) sphere

The red vector is the position vector ($\vec{r}$) for a point on the gaussian sphere.

The blue vector is $\vec{E}_2$.

Let $\vec{d}$ be the vector that joins the point on gaussian sphere with center of spherical hole: $\vec{r} + \vec{d} = \vec{b}$.

$\vec{E}_2$ is in the direction of $\vec{d}$.



Red vector is $\vec{r}$.

Blue vector is $\vec{E}_2$.

$\vec{E}_2$ is in direction of $\vec{b} - \vec{r}$, which is also direction of $\vec{d}$.

Radius of gaussian sphere $|\vec{d}\,| = |\vec{b} - \vec{r}\,|$

Want E_2, electric field of sphere of radius a, charge density ρ

Using previous result, <u>outside sphere radius a</u>, we can pretend that all the charge is at the center

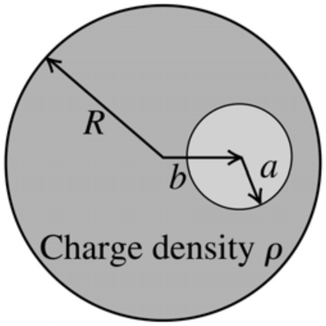
$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\,\frac{q}{d^2}\,\hat{d} = \frac{1}{4\pi\epsilon_0}\,\frac{q}{d^3}\,\vec{d}$$

$$q = \frac{4\pi b^3 \rho}{3} \longrightarrow \vec{E}_2 = \frac{b^3}{3\epsilon_0 d^3}\,\rho\vec{d}$$

On the other hand, inside the hole:

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\,\frac{q}{b^3}\,\vec{d} = \frac{\rho}{3\epsilon_0}\vec{d}$$

Recap of where we are:



- Want field anywhere for r<R
- Trick: add fields from
  1. solid sphere of radius R, q-density +ρ
  2. solid sphere of radius a, q-density -ρ

$$\vec{E}_1(\vec{r}) = \frac{\rho}{3\epsilon_0}\,\vec{r}$$

$$\vec{E}_2(\vec{r}) = \frac{\rho}{3\epsilon_0}\vec{d} \quad \text{or} \quad \vec{E}_2(\vec{r}) = \frac{b^3\rho}{3\epsilon_0}\,\frac{\vec{d}}{d^3}$$

inside the hole

outside the hole

$$\vec{d} = \vec{b} - \vec{r}$$

Now it is simply a matter to adding the two fields:

Inside the hole:

$$\vec{E} = \frac{\rho}{3\epsilon_0}\,\vec{r} \quad + \quad \frac{\rho}{3\epsilon_0}\,(\vec{b} - \vec{r})$$

$$\boxed{\vec{E} = \frac{\rho}{3\epsilon_0}\,\vec{b}}$$

Uniform!

Outside the hole:

$$\boxed{\vec{E} = \frac{\rho}{3\epsilon_0}\,[\vec{r} + \frac{b^3}{|\vec{r}-\vec{b}|^3}(\vec{b} - \vec{r})]}$$