Fall 2004 Physics 3
Tu-Th Section

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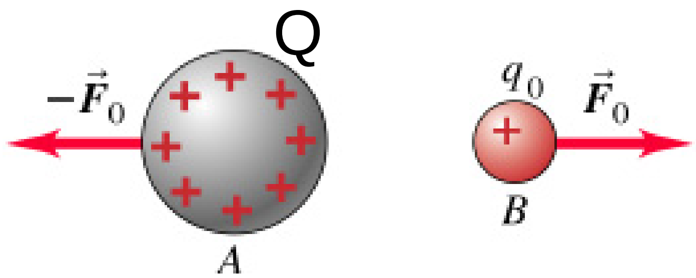
Lecture 6: 12 Oct. 2004

Web page:
http://hep.ucsb.edu/people/claudio/ph3-04/

Electric Field

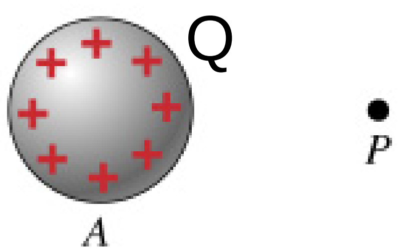
Coulomb force between two charges:



$$F_0 = k\frac{|Qq_0|}{r^2}$$

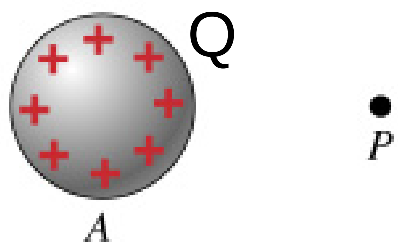
A different picture:

- consider the charge Q all by itself:



If I place a charge q_0 at the point P, this charge will feel a force due to Q

Electric field (cont.)



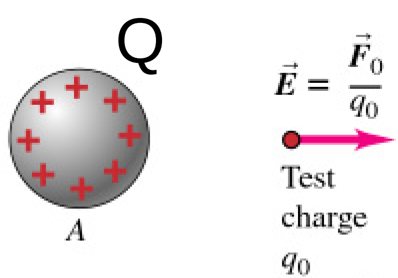
- One way to think about it is this:

  The charge Q somehow modifies the properties of the space around it in such a way that another charge placed near it will feel a force.

- We say that Q generates an "electric field"
- Then a test charge q_0 placed in the electric field will feel a force $\vec{F}_0 = q_0 \vec{E}$

Electric field (vector!)

Test charge at P

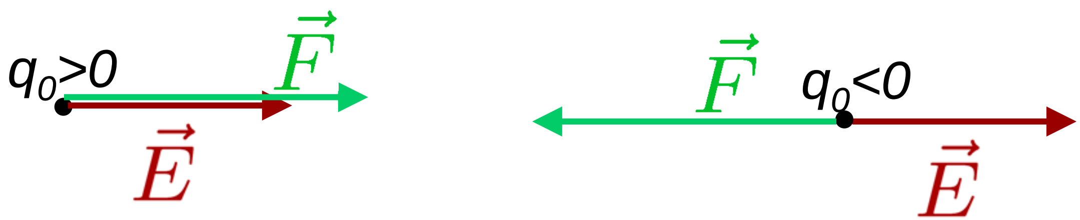


Electric Field Definition

- If a test charge q_0 placed at some point P feels an electric force F_0, then we say that there is an electric field at that point such that:

$$\vec{F}_0 = q_0 \vec{E}$$

- This is a vector equation, both force and electric fields are vectors (have a magnitude and a direction)



- Electric field felt by some charge is created by all other charges.
- Units: Force in N, Charge in C → Electric Field in N/C

Gravitational Field

- The concept of "field" should not be new to you
- Mass *m* near the surface of the earth, then downward force *F=mg* on the mass
- Think of it as $\vec{F} = m\vec{g}$

  where $\vec{g}$ is a gravitational field vector

  - Constant in magnitude and direction (downwards)
- Correspondence

  Electric Field ←→ Gravitational Field

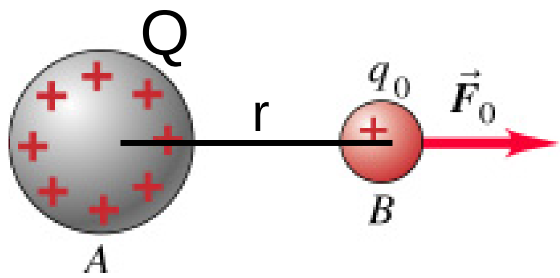
  Eletric charge ←→ Mass

A detail

- Imagine that have some arrangement of charges that creates an electric field
- Now you bring a "test charge" q_0 in
- q_0 will "disturb" the original charges
  - ➢ push them away, or pull them in
- Then the force on q_0 $\vec{F} = q_0 \vec{E}$ will depend on how much the initial charge distribution is disturbed
  - ➢ which in turn depends on how big q_0 is
- This will not do for a definition of E

→ E is defined for an infinitesimally small test charge (limit as $q_0 \to 0$)

Electric Field from a single charge



$$F_0 = k\frac{|Qq_0|}{r^2}$$

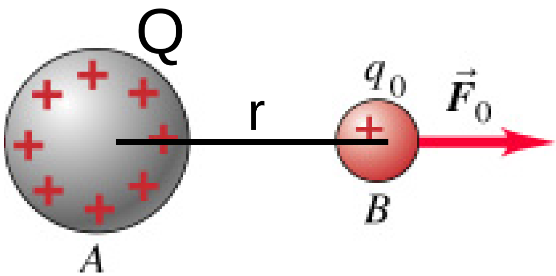
$$\vec{F}_0 = q_0\vec{E}$$

Definition of electric field due to charge *Q* at the point where charge q_0 is placed.

$$\Rightarrow E = k\frac{Q}{r^2}$$

Magnitude of electric field due to *Q* at a distance *r* from *Q*.

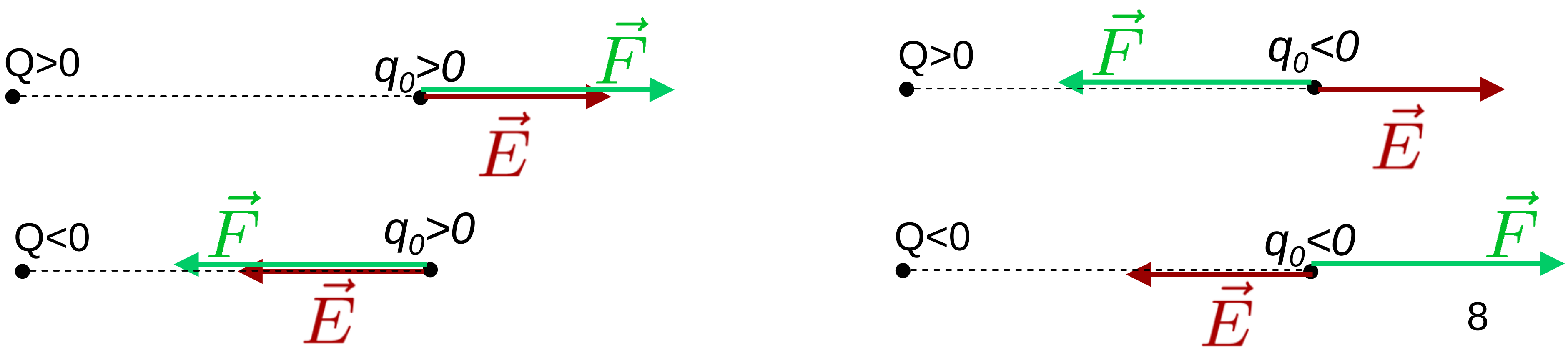
Electric Field from a single charge (cont.)



$$\vec{F}_0 = q_0 \vec{E}$$

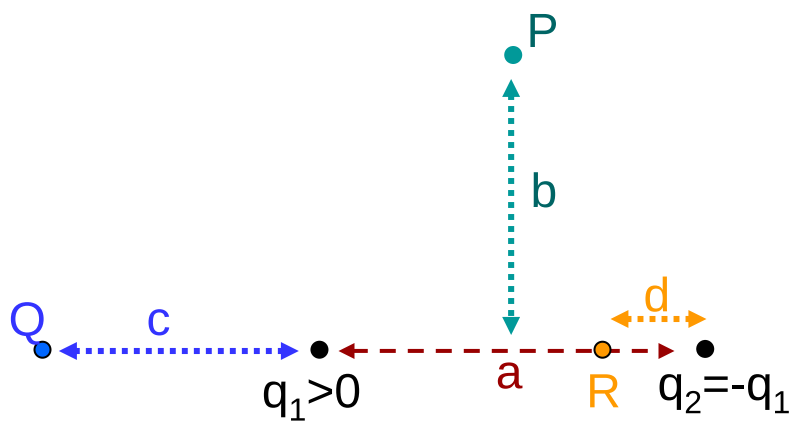
$$E = k\frac{Q}{r^2} \text{ (in magnitude)}$$

- Direction of the electric field at point P?
- Points along the line joining Q with P.
  - If Q>0, points away from Q
  - If Q<0, points towards Q



Example 1 (electric field of a dipole)

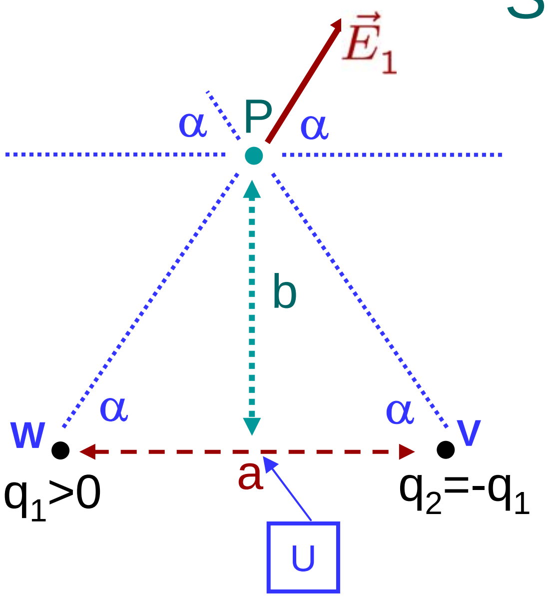
Dipole: a collection of two charges q_1=-q_2

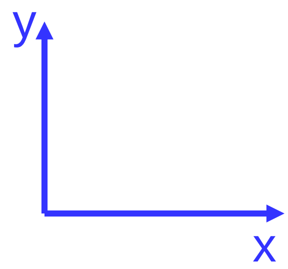


Find the electric field, magnitude and direction at

1. Point P
2. Point Q
3. Point R

Start with point P





Problem setup:

1. Complete labels
   - Label point U,V,W
   - Angle α
2. Choose axes
3. Work out some geometrical relations
   - UW=UV= ½ a
   - UP=UW tanα
     b = ½ a tanα
   - UP=WP sinα
     b = WP sinα
   - UW=WP cosα
     ½ a = WP cosα

VECTOR SUM!

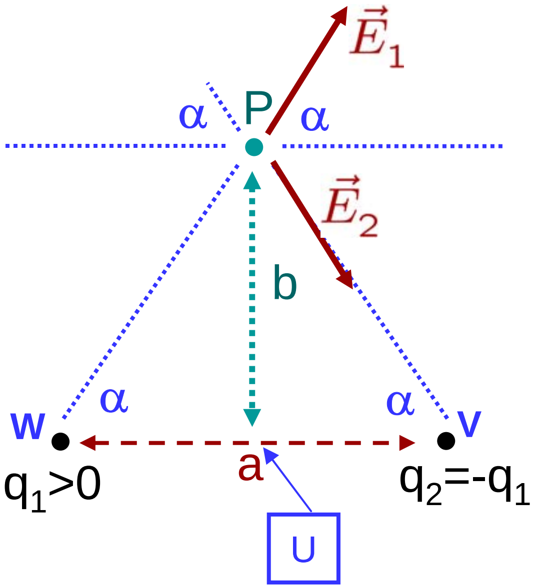
Key concept: Total electric field is the sum of field due to q_1 and field due to q_2

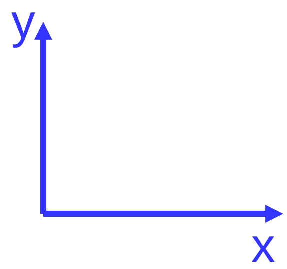
Electric field due to q_1: points away from q_1 because $q_1 > 0$. Call it E_1

Then:

$$E_1 = k\frac{q_1}{|WP|^2} = k\frac{q_1 \sin^2\alpha}{b^2}$$

In components: $E_{1x} = E_1 \cos\alpha$ and $E_{1y} = E_1 \sin\alpha$





$$E_1 = k\frac{q_1}{|WP|^2} = k\frac{q_1 \sin^2\alpha}{b^2}$$

$E_{1x} = E_1 \cos\alpha$ and $E_{1y} = E_1 \sin\alpha$

Now need E_2 = electric field due to q_2

Points <u>towards</u> q_2 (because $q_2 < 0$)

Symmetry:

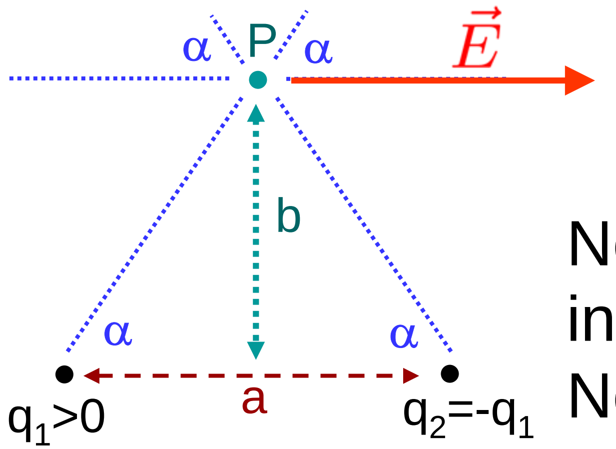
- $|q_1| = |q_2|$ and identical triangles PUW and PUV

→ $E_{2x} = E_{1x}$ and $E_{2y} = - E_{1y}$

Total electric field $\vec{E} = \vec{E}_1 + \vec{E}_2$

→ $E_y = 0$ and $E_x = 2E_{1x} = 2E_1 \cos\alpha$

$$E = E_x = 2k\frac{q_1 \sin^2\alpha \cos\alpha}{b^2}$$



$$E = E_x = 2k\frac{q_1 \sin^2\alpha\cos\alpha}{b^2}$$

Now need to express $\sin^2\alpha$ and $\cos\alpha$ in terms of stuff that we know, i.e., a and b. Note that I do everything with symbols!!

We had

$$b = \tfrac{1}{2}a\tan\alpha = \tfrac{1}{2}a\frac{\sin\alpha}{\cos\alpha}$$

$$\rightarrow \sin\alpha = 2\frac{b}{a}\cos\alpha$$

$$\sin^2\alpha\cos\alpha = 4\frac{b^2}{a^2}\cos^3\alpha$$

Also, trig identity:

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha} = \frac{1}{1+(\frac{2b}{a})^2} = \frac{a^2}{a^2+4b^2}$$

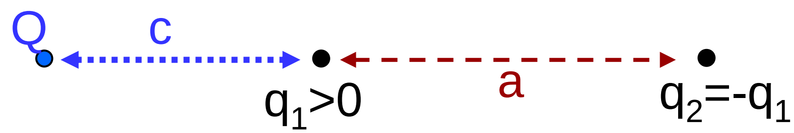
$$\cos^3\alpha = \frac{a^3}{(a^2+4b^2)^{\frac{3}{2}}}$$

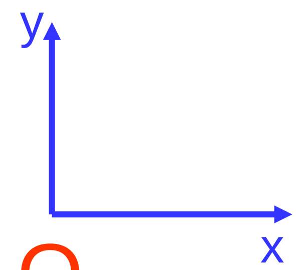
$$\sin^2\alpha\cos\alpha = 4\frac{b^2}{a^2}\cos^3\alpha = 4\frac{b^2}{a^2}\frac{a^3}{(a^2+4b^2)^{\frac{3}{2}}} = \frac{4ab^2}{(a^2+4b^2)^{\frac{3}{2}}}$$

$$E = E_x = 2k\frac{q_1 \sin^2\alpha \cos\alpha}{b^2}$$

$$\sin^2\alpha \cos\alpha = \frac{4ab^2}{(a^2+4b^2)^{\frac{3}{2}}}$$

$$\boxed{E = E_x = 8k\frac{q_1 a}{(a^2+4b^2)^{\frac{3}{2}}}}$$

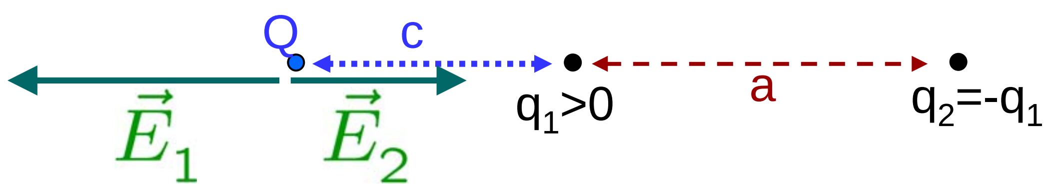




2. Now want electric field at point Q

E_1 due to charge q_1 points away from q_1 ($q_1>0$)

E_2 due to charge q_2 points towards q_2 ($q_2<0$)

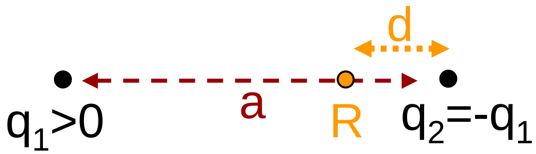


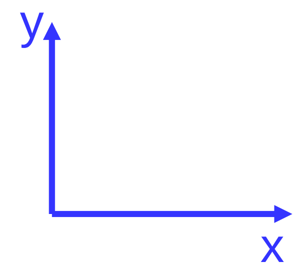
Total electric field $\vec{E} = \vec{E}_1 + \vec{E}_2$

There are no y-components. $E_x = E_{1x} + E_{2x}$

$$E_{1x} = -k\frac{q_1}{c^2} \text{ and } E_{2x} = +k\frac{q_1}{(a+c)^2}$$

$$E_x = kq_1\left(\frac{-1}{c^2} + \frac{1}{(a+c)^2}\right) \quad (<0)$$

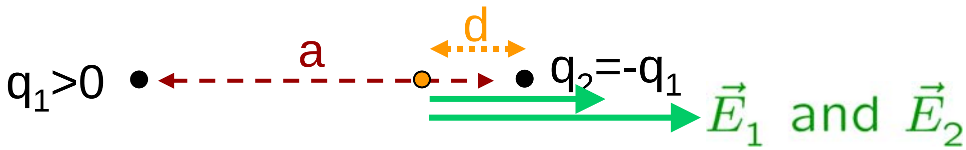




3. Now want electric field at point R

E_1 due to charge q_1 points away from q_1 ($q_1>0$)

E_2 due to charge q_2 points towards q_2 ($q_2<0$)



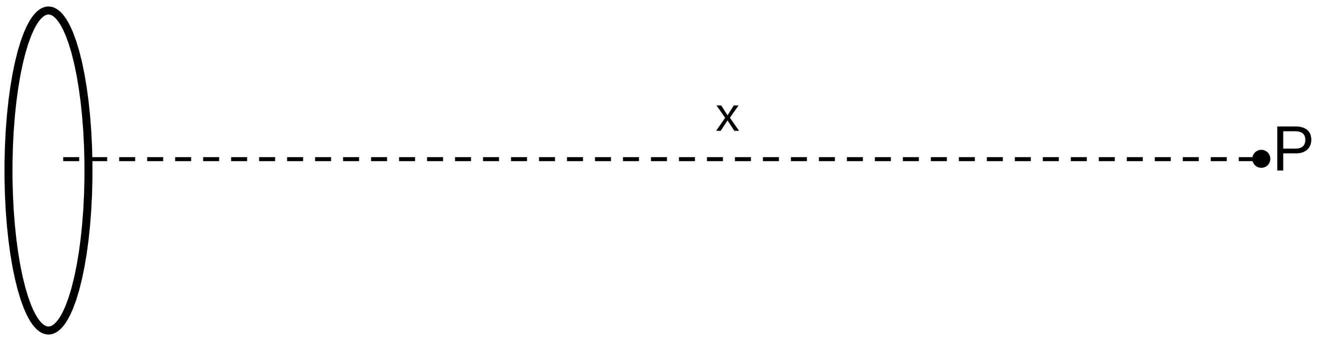
Total electric field $\vec{E} = \vec{E}_1 + \vec{E}_2$

There are no y-components. $E_x = E_{1x} + E_{2x}$

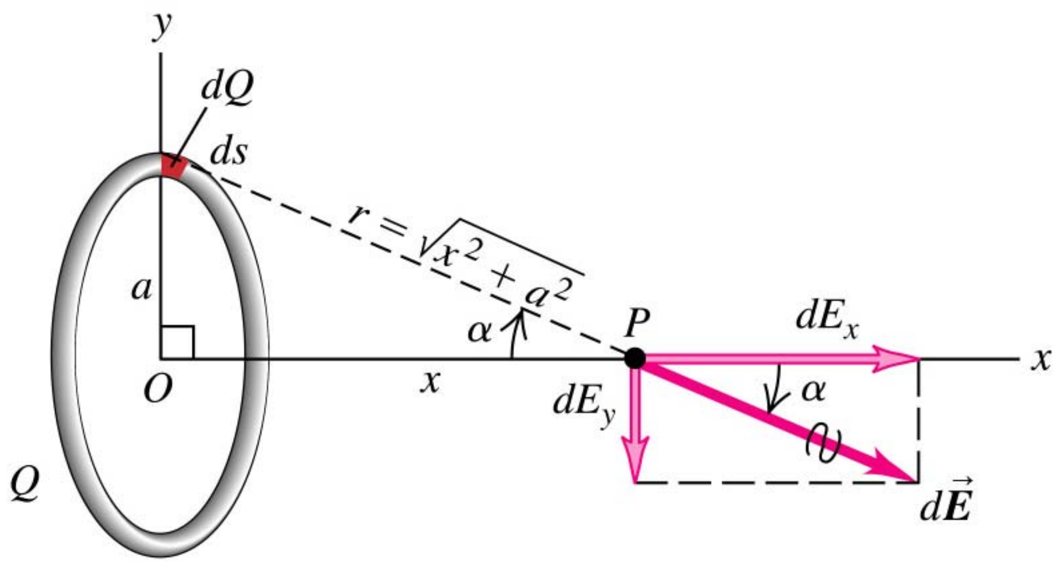
$E_{1x} = +k\frac{q_1}{(a-d)^2}$ and $E_{2x} = +k\frac{q_1}{d^2}$

$$E_x = kq_1\left(\frac{1}{(a-d)^2} + \frac{1}{d^2}\right)$$

Example 2 (field of a ring of charge)



- Uniformly charged ring, total charge *Q*, radius *a*
- What is the electic field at a point P, a distance *x*, on the axis of the ring.
- How to solve
  - Consider one little piece of the ring
  - Find the electric field due to this piece
  - Sum over all the pieces of the ring (<u>VECTOR</u> SUM!!)



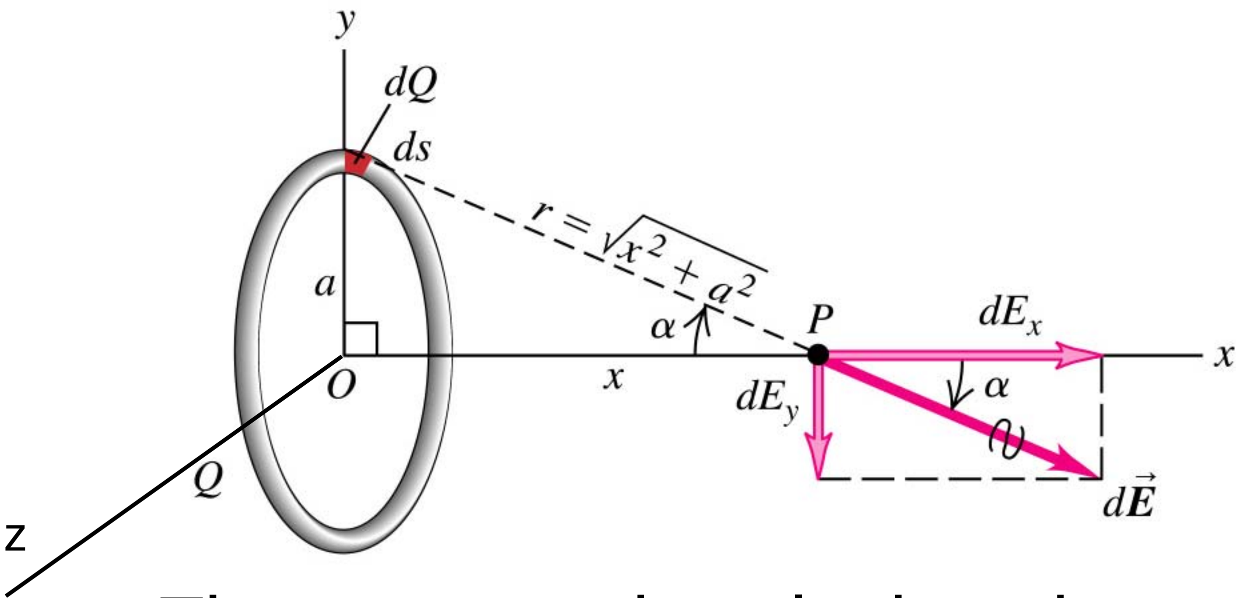
$d\vec{E}$ = electric field due to a small piece of the ring of length ds

dQ = charge of the small piece of the ring

Since the circumference is 2πa, and the total charge is Q:

dQ = Q (ds/2πa)

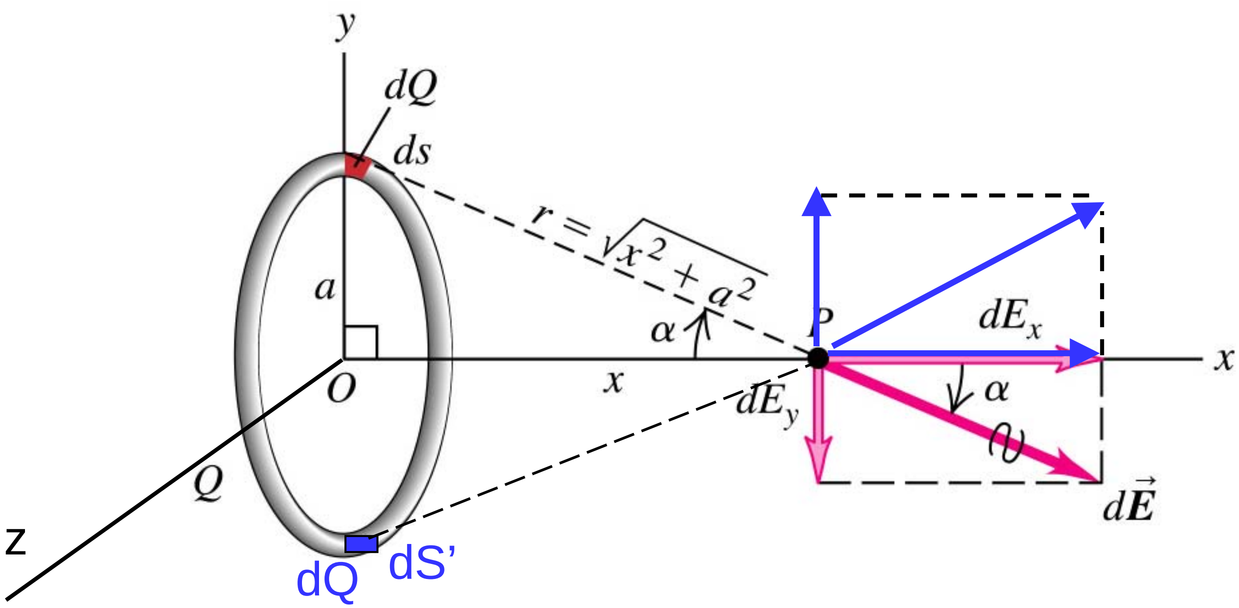
$$dE = k\frac{dQ}{r^2} = k\frac{dQ}{x^2+a^2}$$



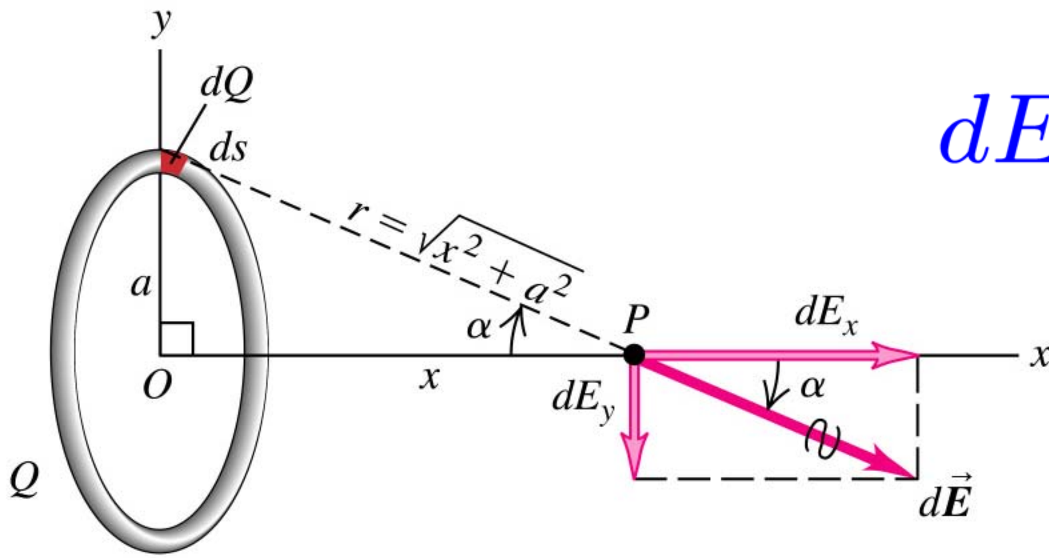
- The next step is to look at the components
- Before we do that, let's think!
  - ➢ We are on the axis of the ring
  - ➢ There cannot be any <u>net</u> y or z components
    - A net y or z component would break the azimuthal symmetry of the problem

→ Let's just add up the x-components and forget about the rest!

What is going on with the y and z components?



The y (or z) component of the electric field caused by the element ds is always exactly cancelled by the electric field caused by the element ds' on the other side of the ring



$$dE = k\frac{Q}{r^2} = k\frac{dQ}{x^2+a^2}$$

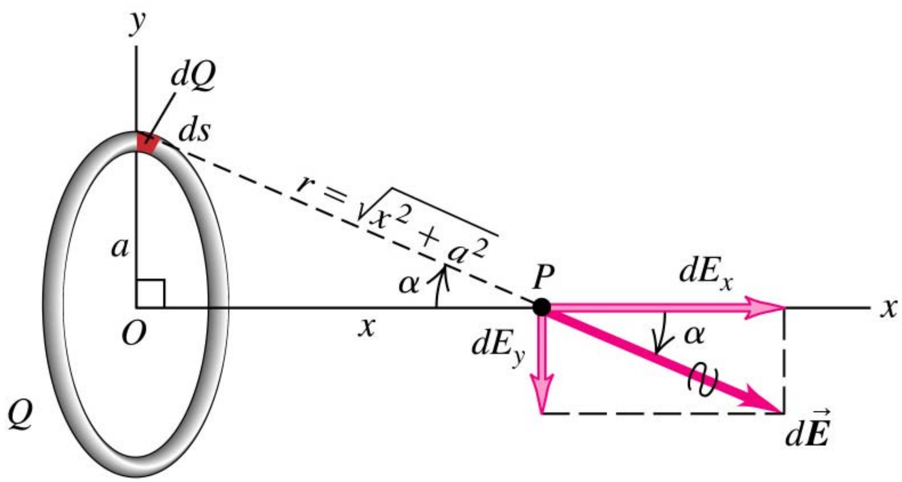
$$dE_x = dE\cos\alpha = k\frac{\cos\alpha \;\; dQ}{x^2+a^2}$$

$$x = r\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2+a^2}}$$

$$dE_x = k\frac{x \;\; dQ}{(x^2+a^2)^{\frac{3}{2}}}$$

Now we sum over the whole ring, i.e. we take the integral:

$$E_x = \int dE_x$$



$$dE_x = \frac{k \ x \ dQ}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$E_x = \int dE_x$$

Time to think about the integral now.
The integration is "over the ring"

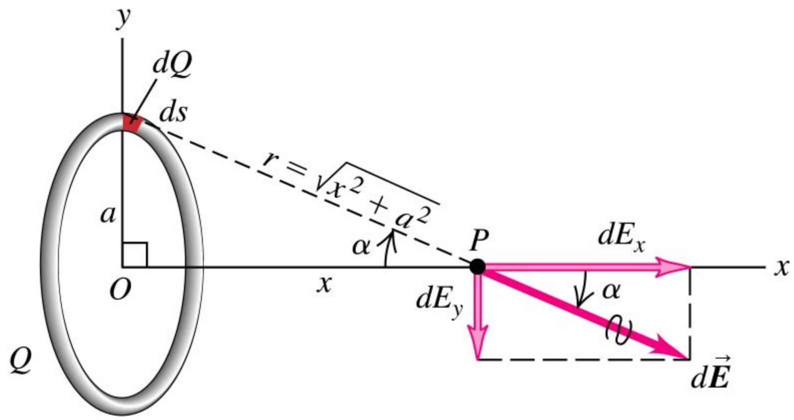
- k is a constant of nature
- a is the ring-radius, a constant for a given ring
- x is the distance from the center of the ring of the point at which we want the E-field,

→ x is also a constant

$$E = \int dE_x = \frac{kx}{(x^2 + a^2)^{\frac{3}{2}}} \int dQ \quad \leftarrow \quad = Q$$

$$E = \frac{kxQ}{(x^2 + a^2)^{\frac{3}{2}}}$$

Sanity check: do limiting cases make sense?

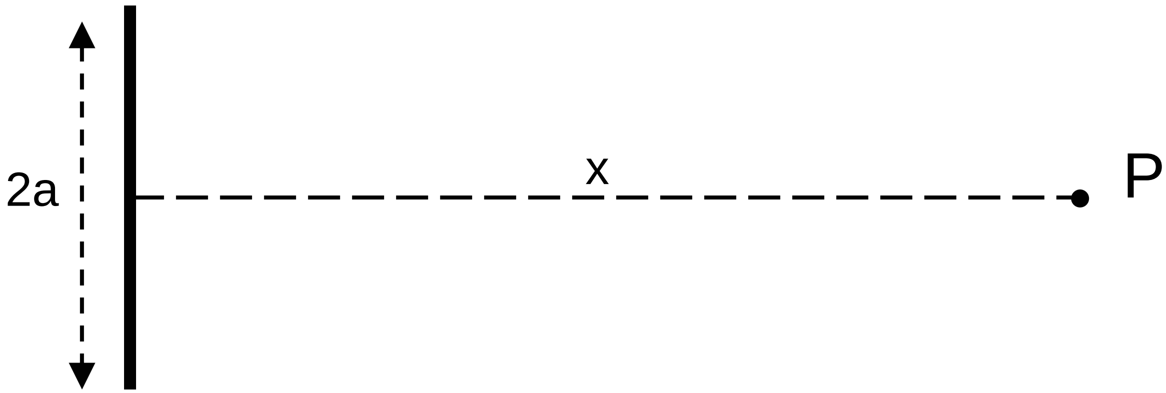


$$E = \frac{kxQ}{(x^2+a^2)^{\frac{3}{2}}}$$

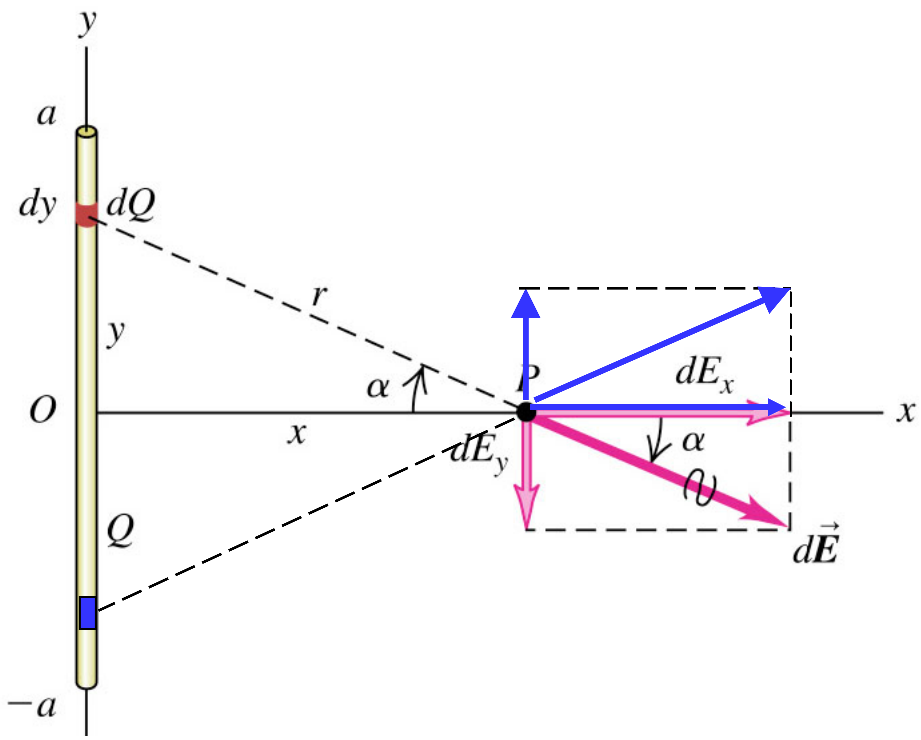
What do we expect for x=0 and x→∞?

- At x=0 expect E=0
  - Again, because of symmetry
  - Our formula gives E=0 for x=0 ☺
- As x→∞, ring should look like a point.
  - Then, should get E→kQ/x^2
  - As x→∞, $(x^2+a^2) \rightarrow x^2$
  - Then E → $kxQ/x^3 = kQ/x^2$ ☺

Example 3 (field of a line of charge)



- Line, length 2a, uniformly charged, total charge Q
- Find the electric field at a point P, a distance x, on axis



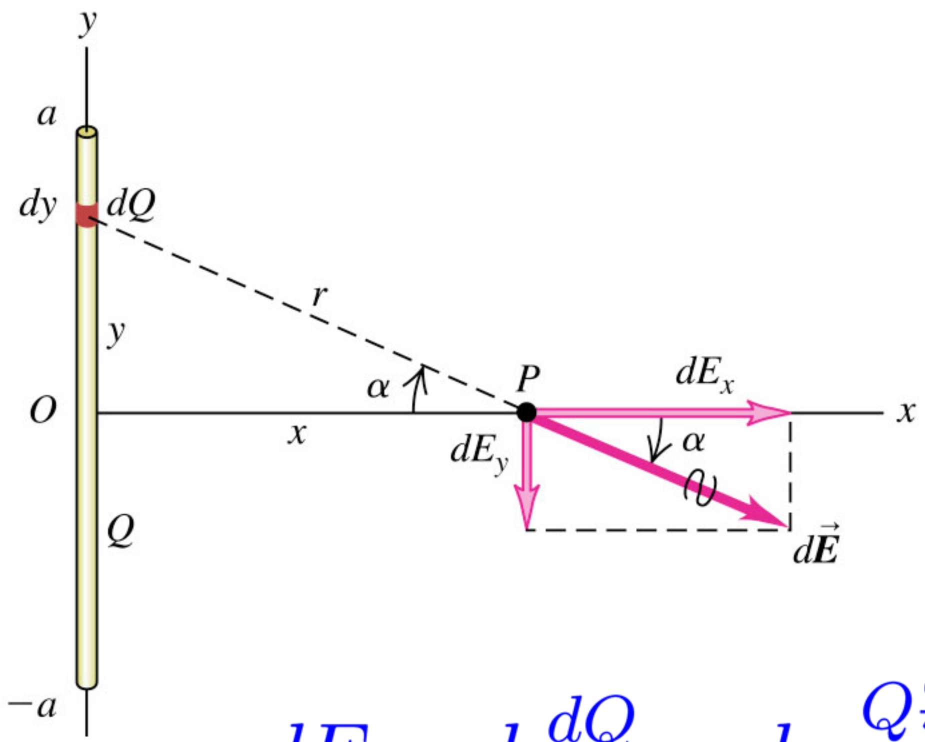
As in the case of the ring, consider field due to small piece (length dy) of the line.

Charge $dQ = Q\ dy/(2a)$

As in the case of the ring, no net y-component

- Because of cancellation from pieces at opposite ends

→ Let's just add up the x-components



$dQ = Q\ (dy/2a)$

$dE_x = dE \cos\alpha$

$y = r \sin\alpha$

$x = r \cos\alpha$

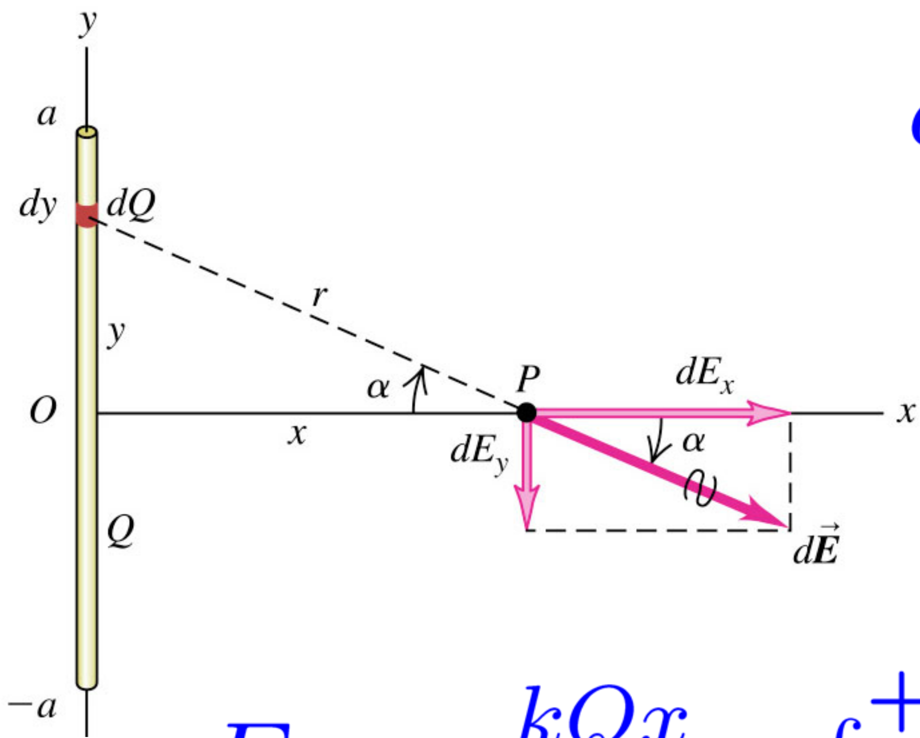
$r^2 = x^2 + y^2$

$$dE = k\frac{dQ}{r^2} = k\frac{Q\frac{dy}{2a}}{x^2+y^2} = \frac{k}{2a}\ \frac{Qdy}{x^2+y^2}$$

$$dE_x = dE\cos\alpha = \frac{k}{2a}\ \frac{Qdy}{x^2+y^2}\cos\alpha$$

$$\cos\alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$dE_x = \frac{kQ}{2a}\ \frac{xdy}{(x^2+y^2)^{\frac{3}{2}}}$$



$$dE_x = \frac{kQ}{2a}\,\frac{x\,dy}{(x^2+y^2)^{\frac{3}{2}}}$$

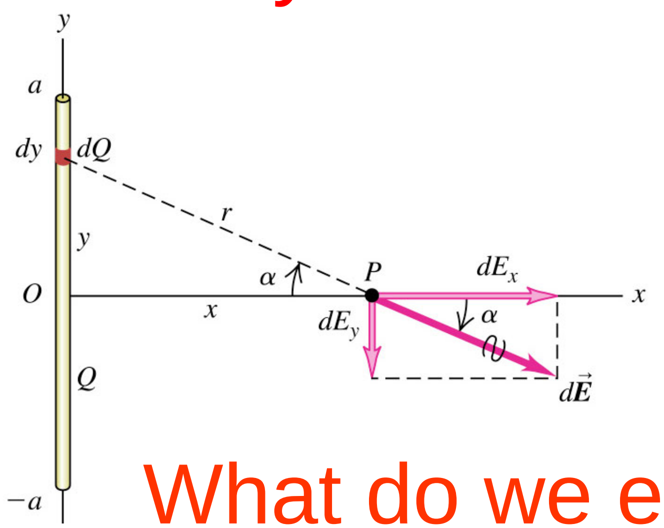
$$E_x = \int dE_x$$

$$E_x = \frac{kQx}{2a}\int_{-a}^{+a}\frac{dy}{(x^2+y^2)^{\frac{3}{2}}}$$

Look up this integral in a table of integrals

$$E_x = k\frac{Q}{x\sqrt{x^2+a^2}}$$

Sanity check: do limiting cases make sense?

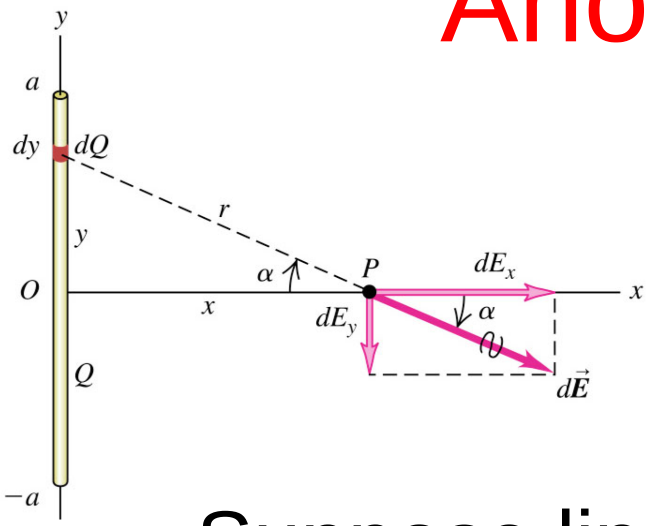


$$E_x = k\frac{Q}{x\sqrt{x^2+a^2}}$$

What do we expect for x→0 and x→∞?

- As x→0 expect E=∞
  - Because at x=0 right "on top" of a charge
  - Our equation works ☺
- As x→∞ line should look like a point
  - Then, should get E→kQ/x^2
  - As x→∞, $(x^2+a^2) \rightarrow x^2$
  - Then E → $kQ/(xx) = kQ/x^2$ ☺

Another limiting case



$$E_x = k\frac{Q}{x\sqrt{x^2+a^2}}$$

- Suppose line is infinitely long ($a \to \infty$)
- Define linear charge density $\lambda = Q/2a$
  - Charge-per-unit-length
- If $a \to \infty$, but x stays finite: $x^2 + a^2 \to a^2$
- Then, denominator → xa

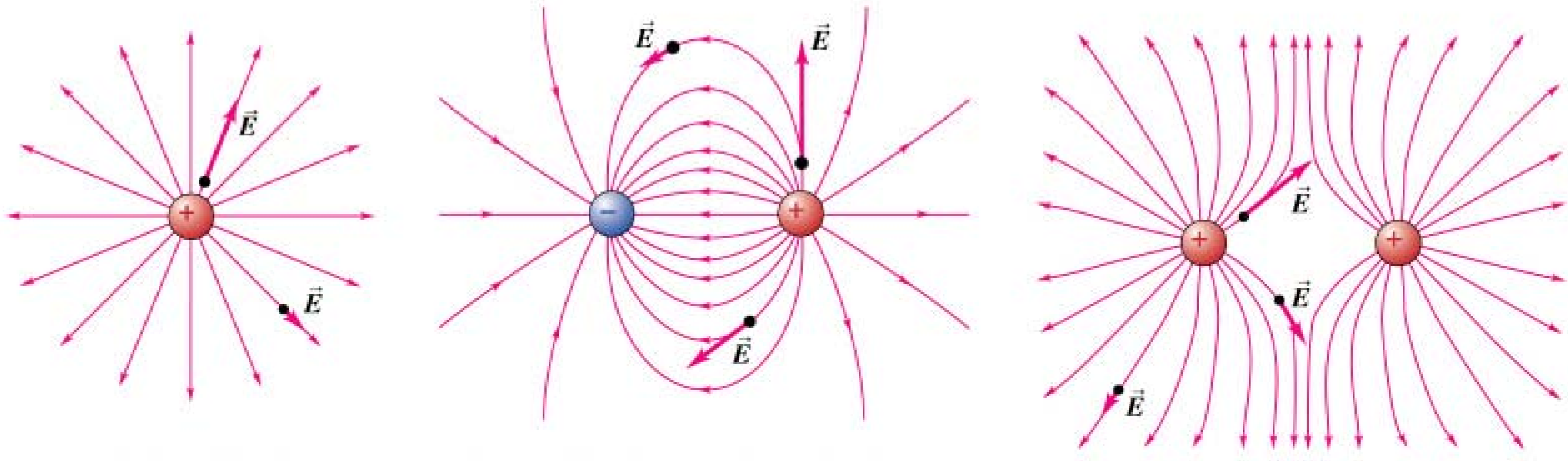
$$E_x \to k\frac{Q}{xa} = k\frac{2\lambda}{x}$$

Jargon and common symbols

- If you have charge on a line (e.g. wire)
  - ➢Linear charge density (λ=Q/L)
    = charge-per-unit-length
- If you have charge on some surface
  - ➢ Surface charge density (σ=Q/A)
    = charge-per-unit-area
- If you have charge distributed in a volume
  - ➢Volume charge density (ρ=Q/V)
    = charge-per-unit-volume

Electric Field Lines

- A useful way to visualize the electric field
- Imaginary lines that are always drawn parallel to the direction of the electric field
- With arrows pointing in the direction of the field



Some properties:

- Lines always start on +ve charges, end on –ve charges
- Density of lines higher where the field is stronger
- Lines never cross
  - Because at each point the field direction is <u>unique</u>