Fall 2004 Physics 3
Tu-Th Section

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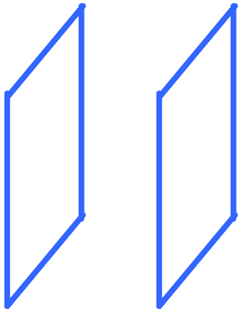
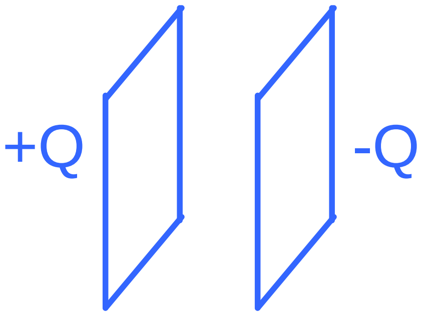
Today: Capacitors

- A capacitor is a device that is used to store electric charge and electric potential energy
- When you look at electric circuits, particularly AC (alternating current) circuits, you will see why this devices are so useful
- A capacitor is a set of two conductors separated by an insulator (or vacuum)
- The "classical" mental picture of a capacitor is two parallel plates



Capacitors, continued

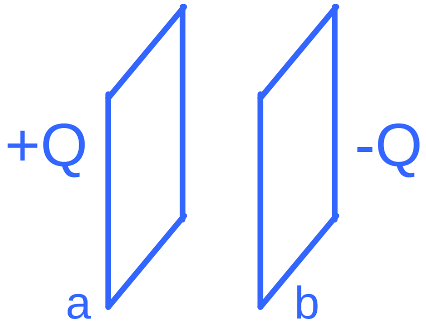
- In the most common situation, the two conductors are initially uncharged and them somehow charge is moved from one to the other.
- Then they have equal and opposite charge



- Q is called <u>the charge on the capacitor</u>

Capacitors (cont.)

- If Q ? 0, there will be a potential difference between the two conductors

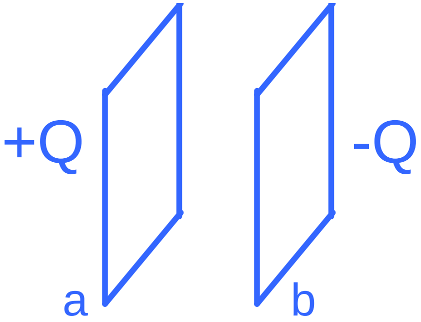


- The capacitance of the system is defined as

$$C = \frac{Q}{V_{ab}}$$

Remember: $V_{ab} = V_a - V_b$

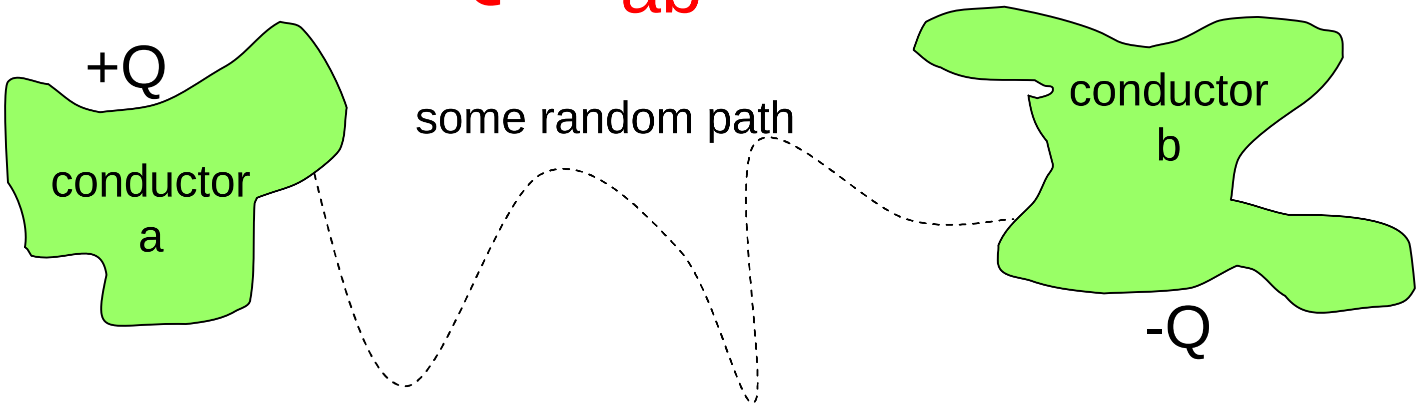
Capacitors (cont.)



$$C = \frac{Q}{V_{ab}}$$

- The capacitance (C) is a property of the conductors
  - Depends on the geometry
    - e.g., for "parallel plate" capacitor, on the surface area of the plates and the distance between them
  - Depends on the material between the two conductors
- $C=Q/V_{ab}$: what does it mean?
  - If I increase V_{ab}, I increase Q
  - If I increase Q, I increase V_{ab}
  - This makes intuitive sense
  - But Q/V_{ab} is a constant
    - Not obvious.

C=Q/V_{ab} Constant



$$V_{ab} = \int_a^b \vec{E} d\vec{l}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_a \frac{dq}{r^2}\hat{r} + \frac{1}{4\pi\epsilon_0}\int_b \frac{dq}{r^2}\hat{r}$$

- If Q doubles (triples, quadruples...), the field doubles (triples, quadruples...)
- Then V_{ab} also doubles (triples, quadruples...)
- But C=Q/V_{ab} remains the same

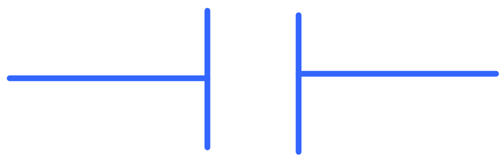
Units of Capacitance

$$C = \frac{Q}{V_{ab}}$$

- [C] = [Charge]/[Voltage] = Coulomb/Volt
- New unit, <u>Farad</u>: 1F = 1 C/V
  - Named after Michael Faraday
- 1 Farad is a huge capacitance
  - We'll see shortly
  - Common units are μF, nF, pF,...

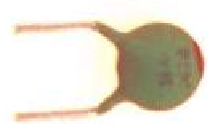
Symbol of capacitance

- The electrical engineers among you will spend a lot of time designing/drawing/struggling-over circuits. In circuits capacitors are denoted by the following symbol

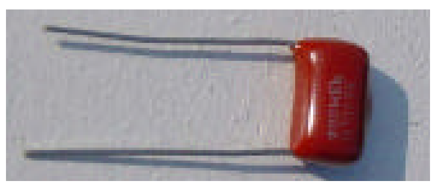


Capacitor types

- Capacitors are often classified by the materials used between electrodes
- Some types are air, paper, plastic film, mica, ceramic, electrolyte, and tantalum
- Often you can tell them apart by the packaging

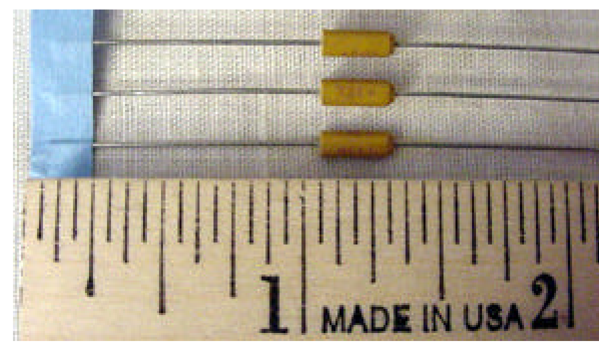


Ceramic Capacitor



Plastic Film Capacitor

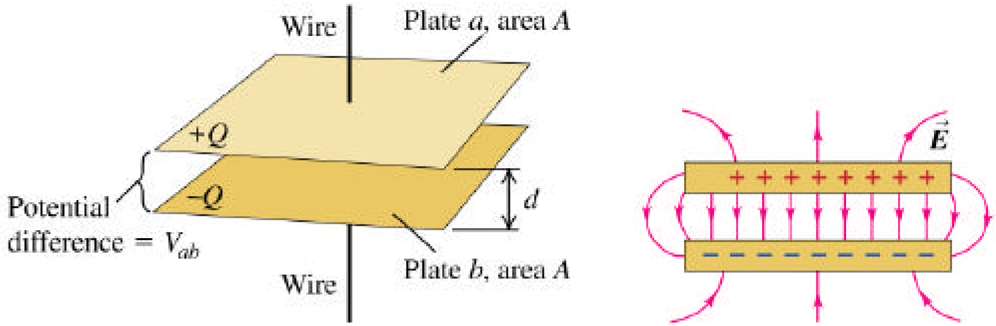
Electrolyte Capacitor



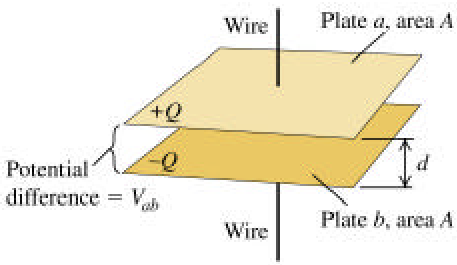
Tantalum Capacitor

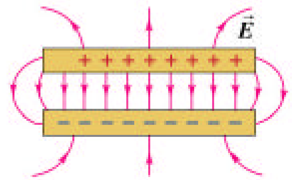
Parallel Plate Capacitor (vacuum)

- Calculate capacitance of parallel plate capacitor with no material (vacuum) between plates



- Ignoring edge effects, the electric field is uniform between the two plates
  - ➢ We showed (Chapter 21) that the electric field between two infinitely large, flat conductors, with surface charge densities $+\sigma$ and $-\sigma$ is $E=\sigma/\varepsilon_0$
- $\sigma = Q/A \rightarrow E=Q/(A\varepsilon_0)$





$E=Q/(A\varepsilon_0)$

$$C = \frac{Q}{V_{ab}}$$

We now want a relationship between E and V_{ab} ($V_{ab}=V_a - V_b$)

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}$$

$$V_{ab} = Ed = Qd/(A\varepsilon_0)$$

$$C = \frac{Q}{V_{ab}} = \epsilon_0 \frac{A}{d}$$

Depends only on the geometry (A and d), as advertised

1 Farad is a huge capacitance!

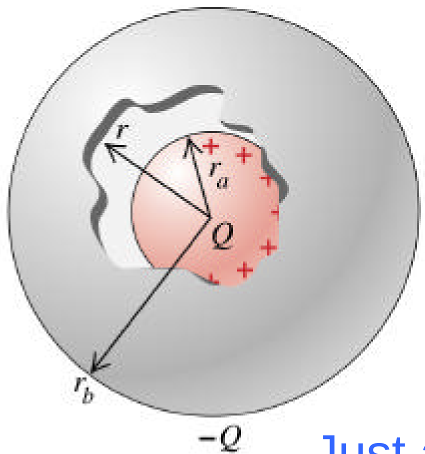
- Take two parallel plates, d=1 mm apart.
- How large must the plates be (in vacuum) for C=1 F?

$$C = \epsilon_0 \frac{A}{d} \quad \rightarrow \quad A = \frac{C \cdot d}{\epsilon_0}$$

$$A = \frac{1\mathrm{F} \cdot 10^{-3}\mathrm{m}}{8.85\ 10^{-12}\mathrm{F/m}} = 1.1\ 10^{8}\mathrm{m}^{2}$$

Pretty large!

Capacitance of a Spherical Capacitor



Two concentric spherical shells.
Radii r_a and r_b

Just as in the problem of the parallel plate capacitor, we will:

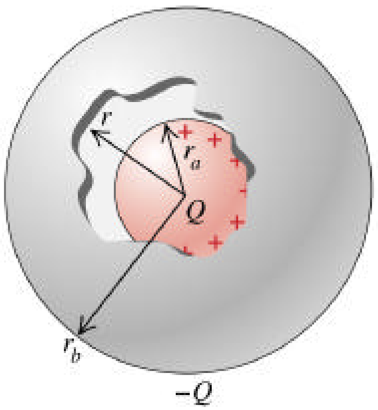
1. Calculate the electric field between the two conductors
2. From the electric field, calculate V_{ab} from $V_{ab} = \int E dl$
3. Take C=Q/V

To calculate the electric field between the two shells we use Gauss's law.
Remember, Gauss's law:

$$\oint \vec{E} d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

In our case, $Q_{enclosed}$=Q
The flux is $(4\pi\, r^2)E$

$$E = \frac{1}{4\pi\epsilon_0}\,\frac{Q}{r^2}$$



Plan was:

1. Calculate the electric field between the two conductors
2. From the electric field, calculate V_{ab} from $V_{ab} = \int E dl$
3. Take C=Q/V

$$1:\ E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

$$2: V_{ab} = \int_a^b E dr = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{Q dr}{r^2}$$

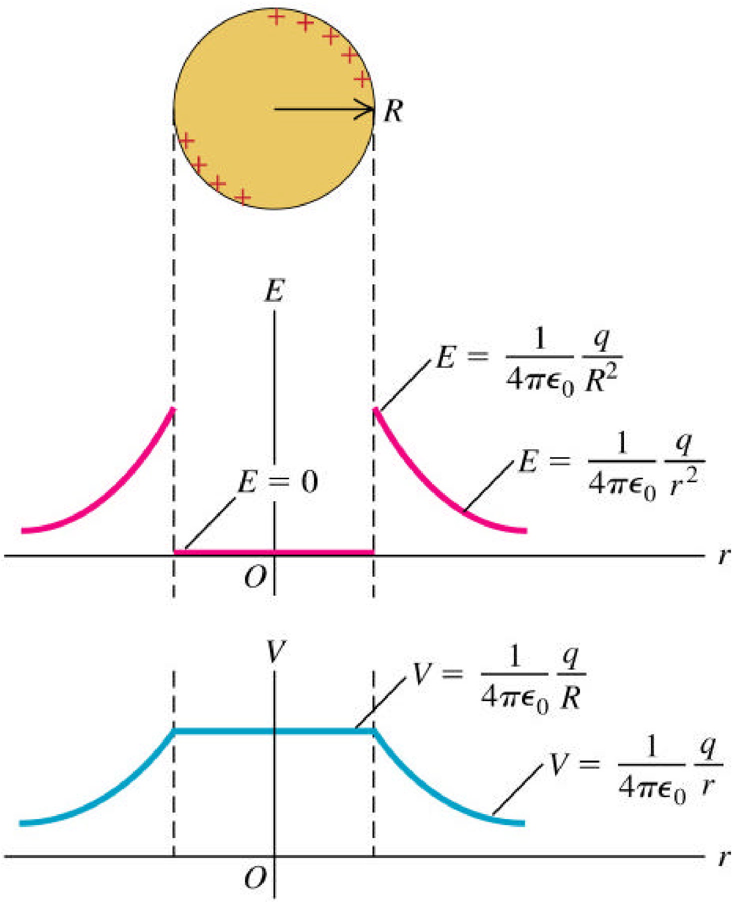
$$V_{ab} = \frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{r_a}^{r_b} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$V_{ab} = \frac{Q}{4\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

$$3: C = \frac{Q}{V_{ab}} \quad \rightarrow \quad C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

Depends only on the geometry, as advertised

- Note: we could have saved ourselves some work!
- In the previous lecture we calculated the potential due to a conducting sphere

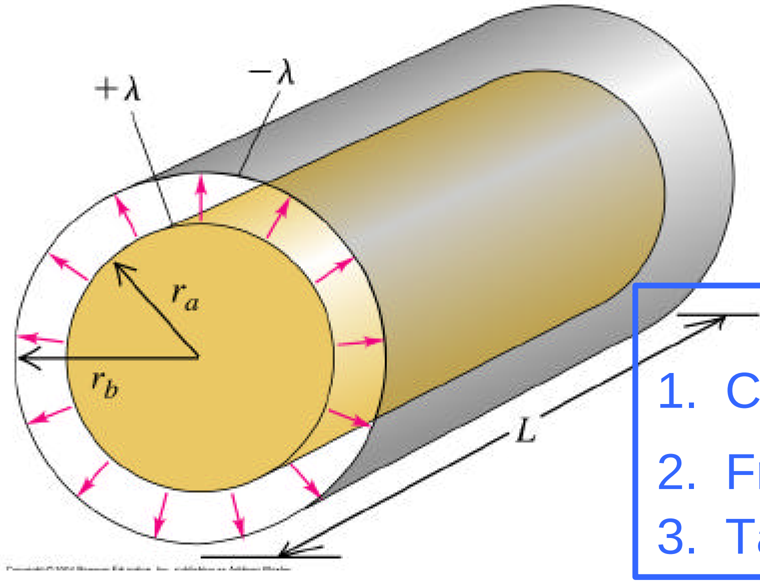


- How could we have used this result, since now we have two concentric shells?
- First, because the charge is on the surface, it does not matter if it is a shell or a sphere
- Second, by Gauss's law the field, and thus the potential depends only on the enclosed charge, i.e. the charge on the inner sphere
- So we could have immediately written

$$V_a = \frac{1}{4\pi\epsilon_0}\ \frac{Q}{r_a}$$

$$V_b = \frac{1}{4\pi\epsilon_0}\ \frac{Q}{r_b}$$

Capacitance of a cylindrical capacitor



Two concentric cylinders
Radii r_a and r_b

Brute force approach:

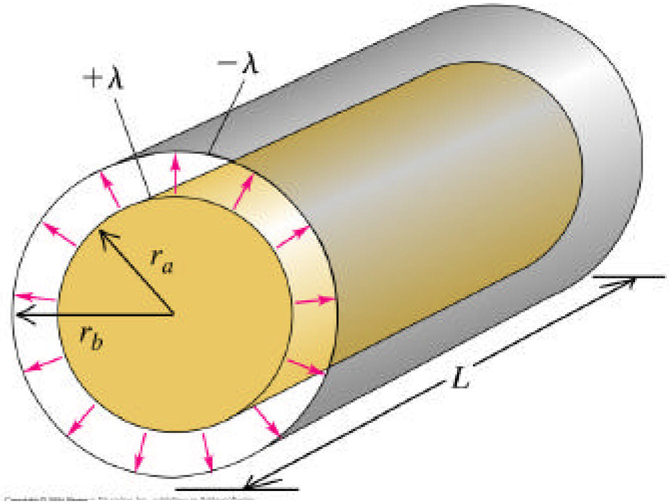
1. Calculate the field between the two conductors
2. From the field, calculate V_{ab} from $V_{ab} = \int E dl$
3. Take C=Q/V

Time saving approach
Use result from previous lecture
Potential due to (infinite) line of charge

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

Why can I use the line of charge result?

1. Because the field (or potential) outside a cylinder is the same as if the charge was all concentrated on the axis
2. Because of Gauss's law the field between the two cylinders is the same as if the outermost cylinder was not there



$$V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

$$C = \frac{Q}{V_{ab}}$$

$$V_{ab} = V_a - V_b = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r_a} - \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r_b}$$

Using log(A/B) = logA – logB, we get

$$V_{ab} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_b}{r_a}$$

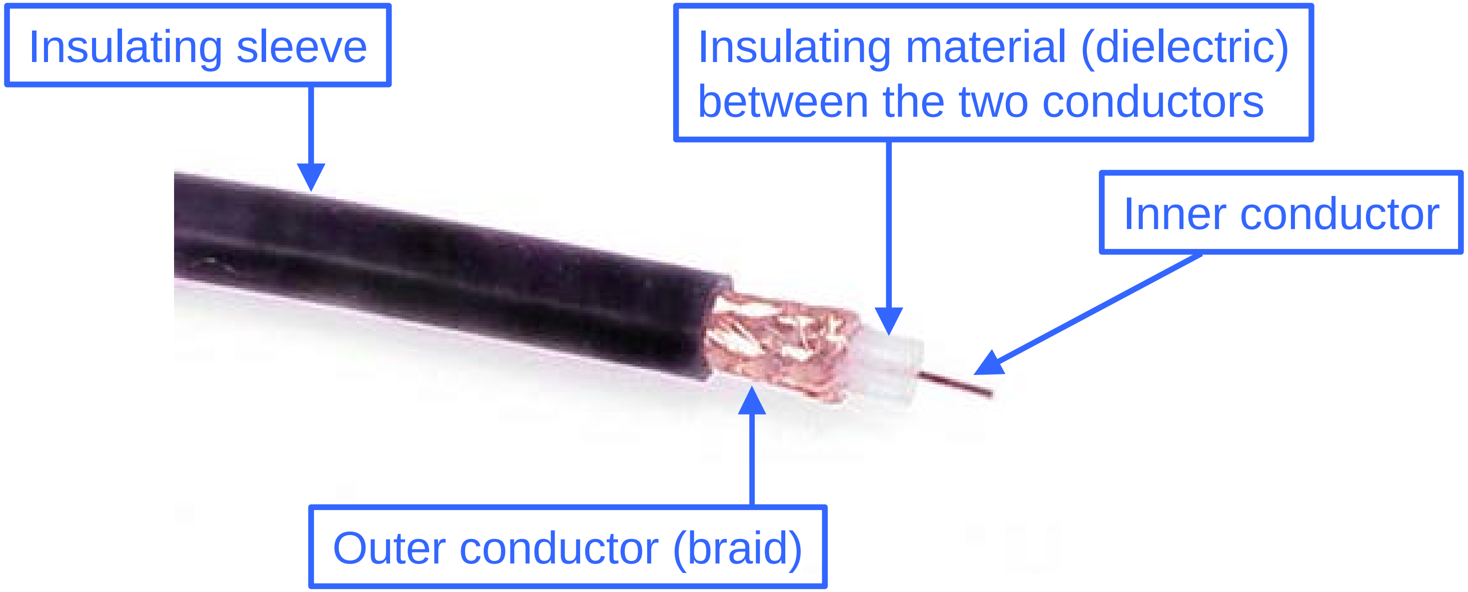
Using λ = Q/L we get $V_{ab} = \frac{Q}{2L\pi\epsilon_0} \ln \frac{r_b}{r_a}$

$$C = \frac{Q}{V_{ab}} \quad \rightarrow \quad C = \frac{2\pi\epsilon_0 L}{\ln(r_b/r_a)}$$

Depends only on the geometry, as advertised

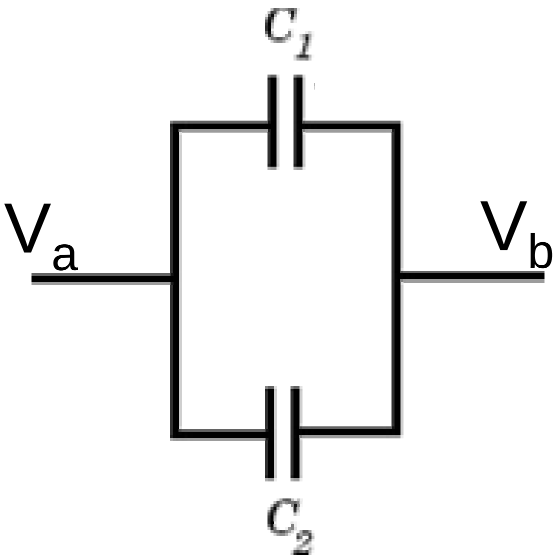
Coaxial Cable

- The cable that you plug into your TV to receive "cable TV" is just like a cylindrical capacitor

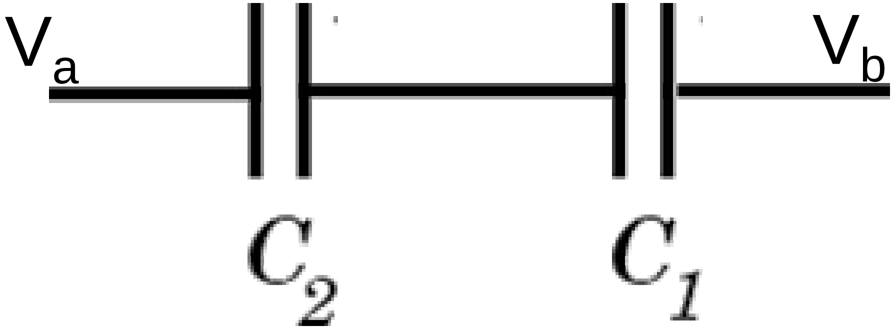


Connecting capacitors together

Two ways of connecting capacitors together:

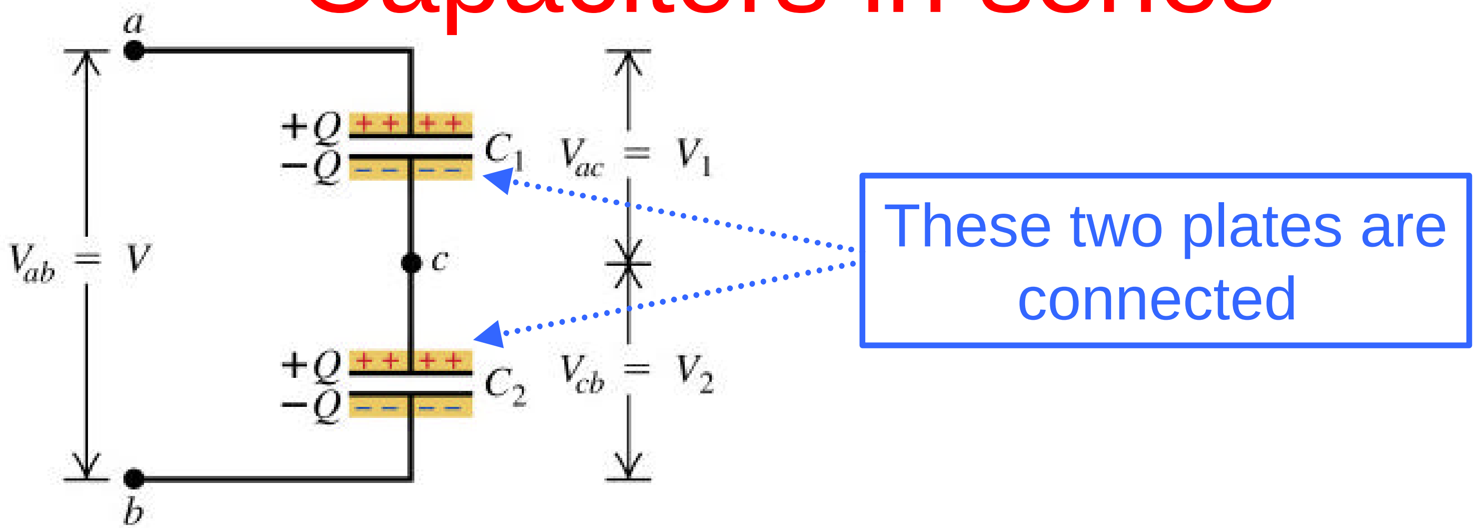


in parallel



in series

Capacitors in series



The two connected plates effectively form a single conductor
Thus, the two connected plates have equal and opposite charge

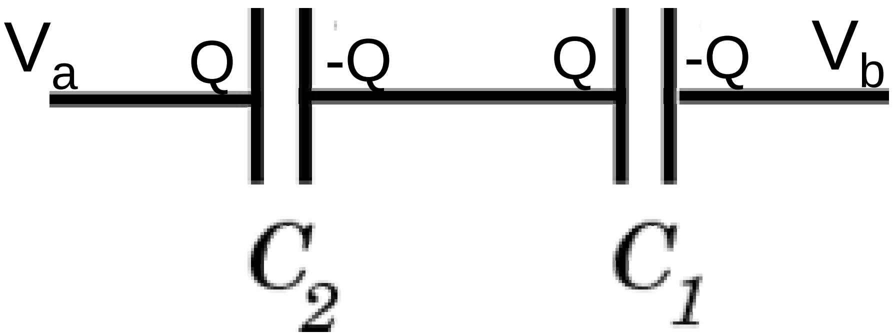
$$V_{ac} = V_a - V_c = \frac{Q}{C_1}$$

$$V_{cb} = V_c - V_b = \frac{Q}{C_2}$$

$$V_{ab} = V_a - V_b = (V_a - V_c) + (V_c - V_b) = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$V_{ab} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

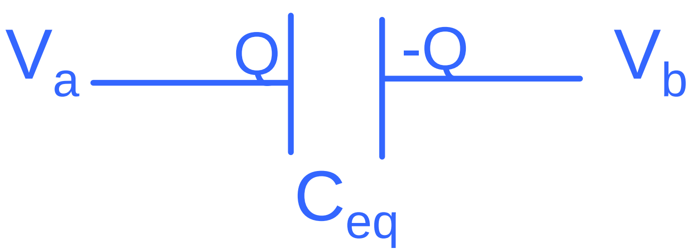
Capacitors in series (cont.)



$$V_{ab} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

Remember, definition: $C = \dfrac{Q}{V_{ab}}$

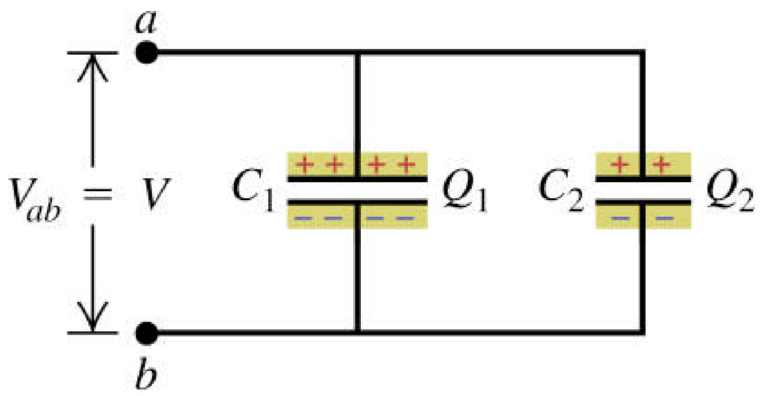
Thus, this is <u>entirely</u> equivalent to



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

equivalent capacitance

Capacitors in parallel

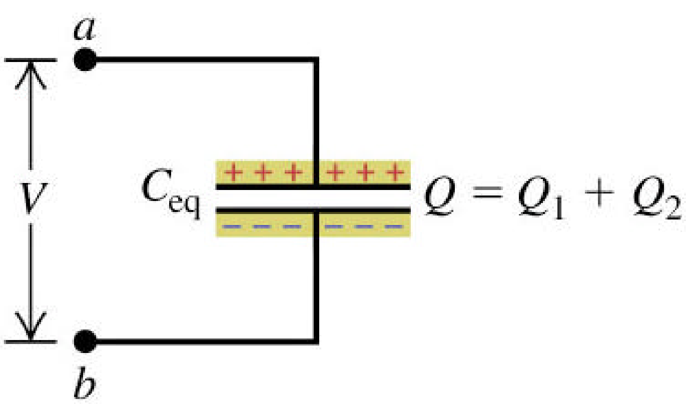


The potential difference across the two capacitors is the same

$Q_1 = C_1V_{ab}$ and $Q_2 = C_2\,V_{ab}$

Therefore, $Q=Q_1+Q_2 = (C_1 + C_2)\,V_{ab}$

This is equivalent to



$$C_{eq} = C_1 + C_2$$

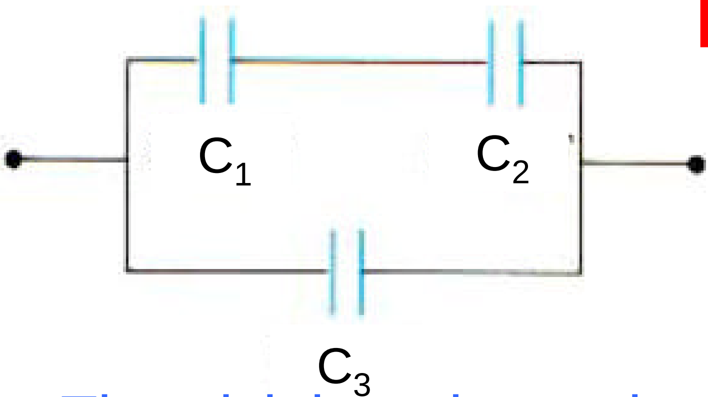
equivalent capacitance

For more than two capacitors in parallel or in serees the results generalize to

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \qquad \text{(capacitors in series)}$$

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \qquad \text{(capacitors in parallel)}$$

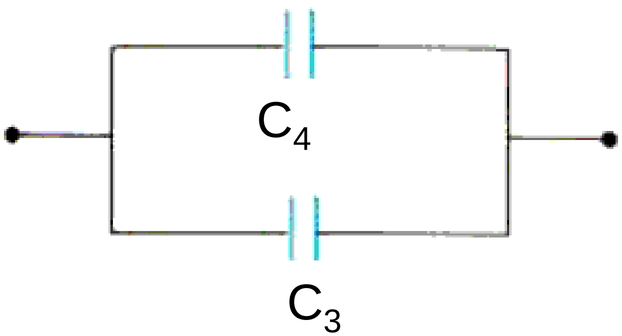
Example



Find the equivalent capacitance of this network.

The trick here is to take it one step at a time

C_1 and C_3 are in series. So this circuit is equivalent to

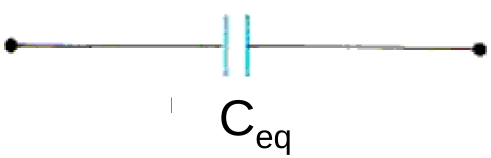


$$\frac{1}{C_4} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_4 = \frac{C_1 C_2}{C_1 + C_2}$$

Then, this is equivalent to

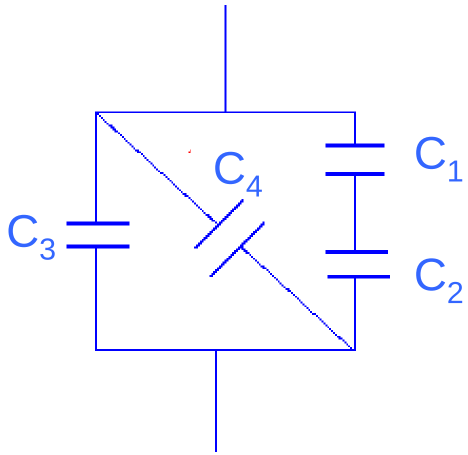
$$C_{eq} = C_3 + C_4$$



$$C_{eq} = C_3 + \frac{C_1 C_2}{C_1 + C_2}$$

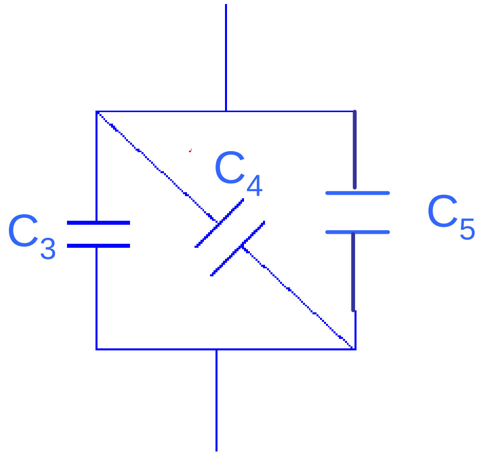
$$C_{eq} = \frac{C_1 C_2 + C_1 C_3 + C_2 C_3}{C_1 + C_2}$$

Another example



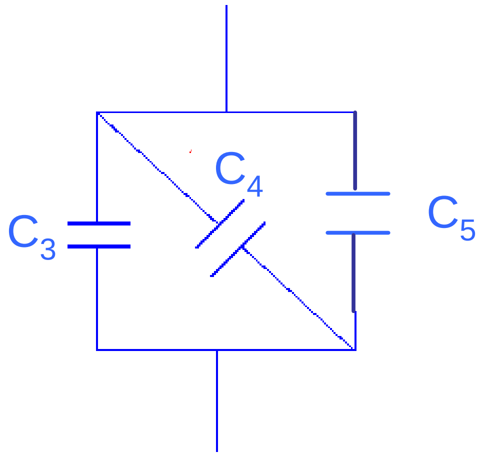
Find the equivalent capacitance of this network.

Again, take it in steps. C_1 and C_2 are in series. So this is equivalent to



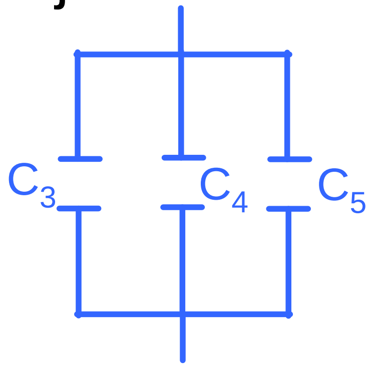
$$\frac{1}{C_5} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_5 = \frac{C_1 C_2}{C_1 + C_2}$$

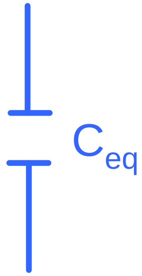


$$C_5 = \frac{C_1 C_2}{C_1 + C_2}$$

Now this looks a little different than what we have seen.
But it is just three capacitors in parallel. We can redraw it as



which is equivalent to



$$C_{eq} = C_3 + C_4 + C_5 = C_3 + C_4 + \frac{C_1 C_2}{C_1 + C_2}$$

$$\boxed{C_{eq} = \frac{C_1 C_2 + C_1 C_3 + C_2 C_3 + C_1 C_4 + C_2 C_4}{C_1 + C_2}}$$

Energy stored in a capacitor

- A capacitor stores potential energy
- By conservation of energy, the stored energy is equal to the work done in charging up the capacitor
- Our goal now is to calculate this work, and thus the amount of energy stored in the capacitor

- Once the capacitor is charged $V = V_{ab} = \frac{Q}{C}$
  - Let q and v be the charge and potential of the capacitor at some instant while it is being charged
    - q<Q and v<V, but still v=q/C
  - If we want to increase the charge from q → q+dq, we need to do an amount of work dW

$$dW = vdq = \frac{qdq}{C}$$

- The total work done in charging up the capacitor is

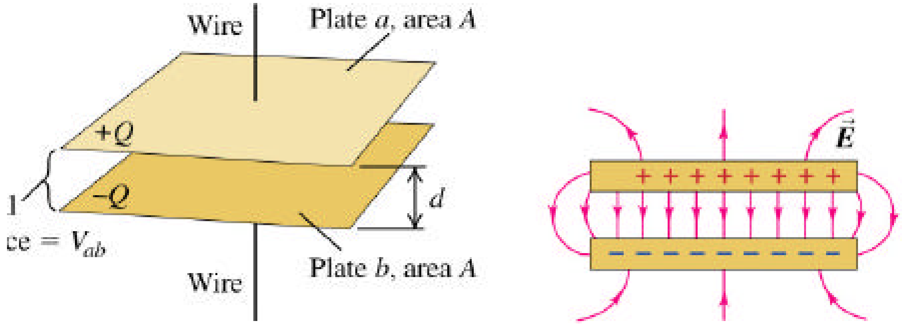
$$W = \int_0^Q \frac{qdq}{C} = \frac{Q^2}{2C}$$

- Potential energy stored in the capacitor is

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Energy in the electric field

- If a capacitor is charged, there is an electric field between the two conductors



- We can think of the energy of the capacitor as being stored in the electric field
- For a parallel plate capacitor, ignoring edge effects, the volume over which the field is active is Axd

- Then, the energy per unit volume (energy density) is

$$u = \frac{\frac{1}{2}CV^2}{Ad}$$

- But the capacitance and electric field are given by

$$C = \epsilon_0 \frac{A}{d} \qquad E = \frac{V}{d}$$

- Putting it all together:

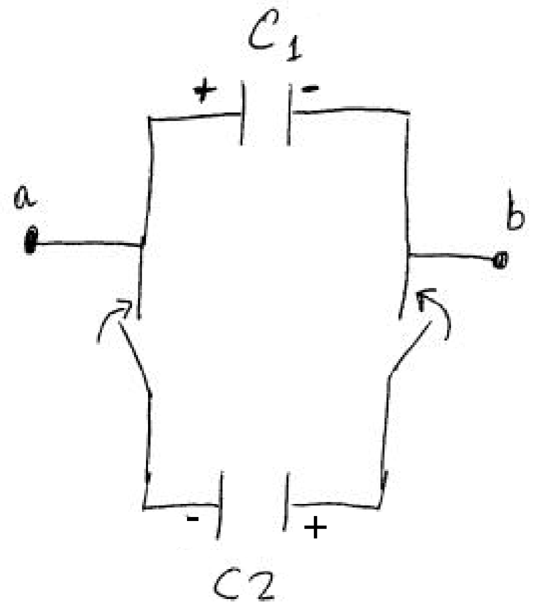
$$u = \frac{\frac{1}{2}\epsilon_0 \frac{A}{d}(Ed)^2}{Ad}$$

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$

- This is the energy density (energy per unit volume) associated with an electric field
  - ➢ Derived it for parallel plate capacitor, but valid in general

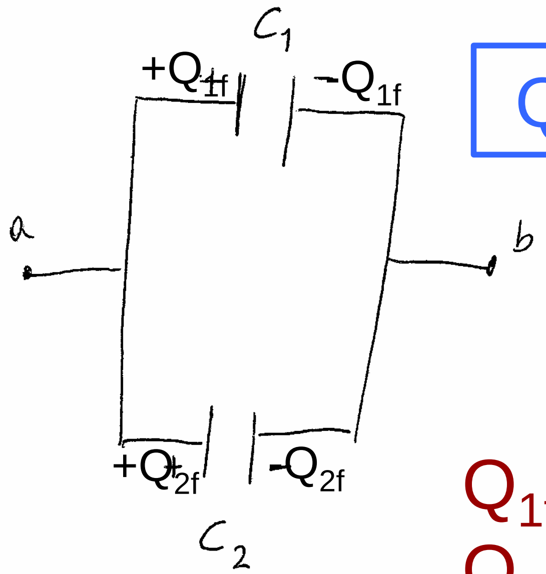
Example

- C_1 and C_2 (C_1>C_2) are both charged to potential V, but with opposite polarity. They are removed from the battery, and are connected as shown. Then we close the two switches. Find V_{ab} after the switches have been closed



Q_{1i} = initial charge of C_1 = C_1V
Q_{2i} = initial charge of C_2 = - C_2V
→Charge $Q_{total} = Q_{1i} + Q_{2i} = (C_1-C_2)V$

After we close the switches, this charge will distribute itself partially on C_1 and partially on C_2, but with $Q_{total} = Q_{1f} + Q_{2f}$



$$Q_{total} = Q_{1i} + Q_{2i} = (C_1 - C_2)V = Q_{1f} + Q_{2f}$$

$$Q_{1f} = C_1 V_{ab}$$
$$Q_{2f} = C_2 V_{ab}$$

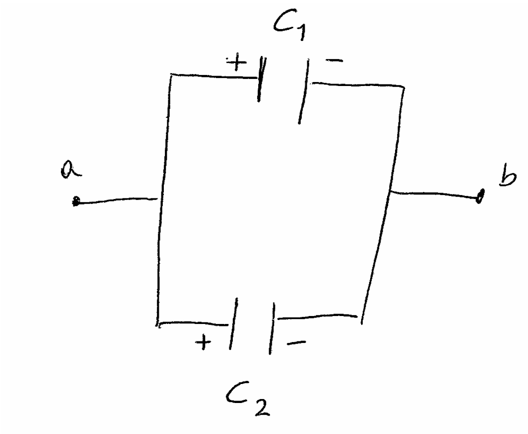
$\rightarrow$ $$Q_{1f} + Q_{2f} = (C_1 + C_2) V_{ab}$$

Then, equating the two boxed equations

$$V_{ab} = V \frac{C_1 - C_2}{C_1 + C_2}$$

Now calculate the energy before and after

- $E_{before} = ½ C_1 V^2 + ½ C_2 V^2 = ½ (C_1 + C_2) V^2$
- $E_{after} = ½ C_{eq} V_{ab}$, where C_{eq} is the equivalent capacitance of the circuit after the switches have been closed



- C_1 and C_2 are in parallel

→ $C_{eq} = C_1 + C_2$

→ $E_{after} = ½ (C_1 + C_2) V_{ab}$

$$\frac{E_{\text{after}}}{E_{\text{before}}} = \frac{\frac{1}{2}(C_1+C_2)V_{ab}^2}{\frac{1}{2}(C_1+C_2)V^2} = \frac{V_{ab}^2}{V^2}$$

$$\frac{E_{\text{after}}}{E_{\text{before}}} = \left(\frac{C_1-C_2}{C_1+C_2}\right)^2 < 1$$

What happens to conservation of energy????

It turns out that some of the energy is radiated as electromagnetic waves!!