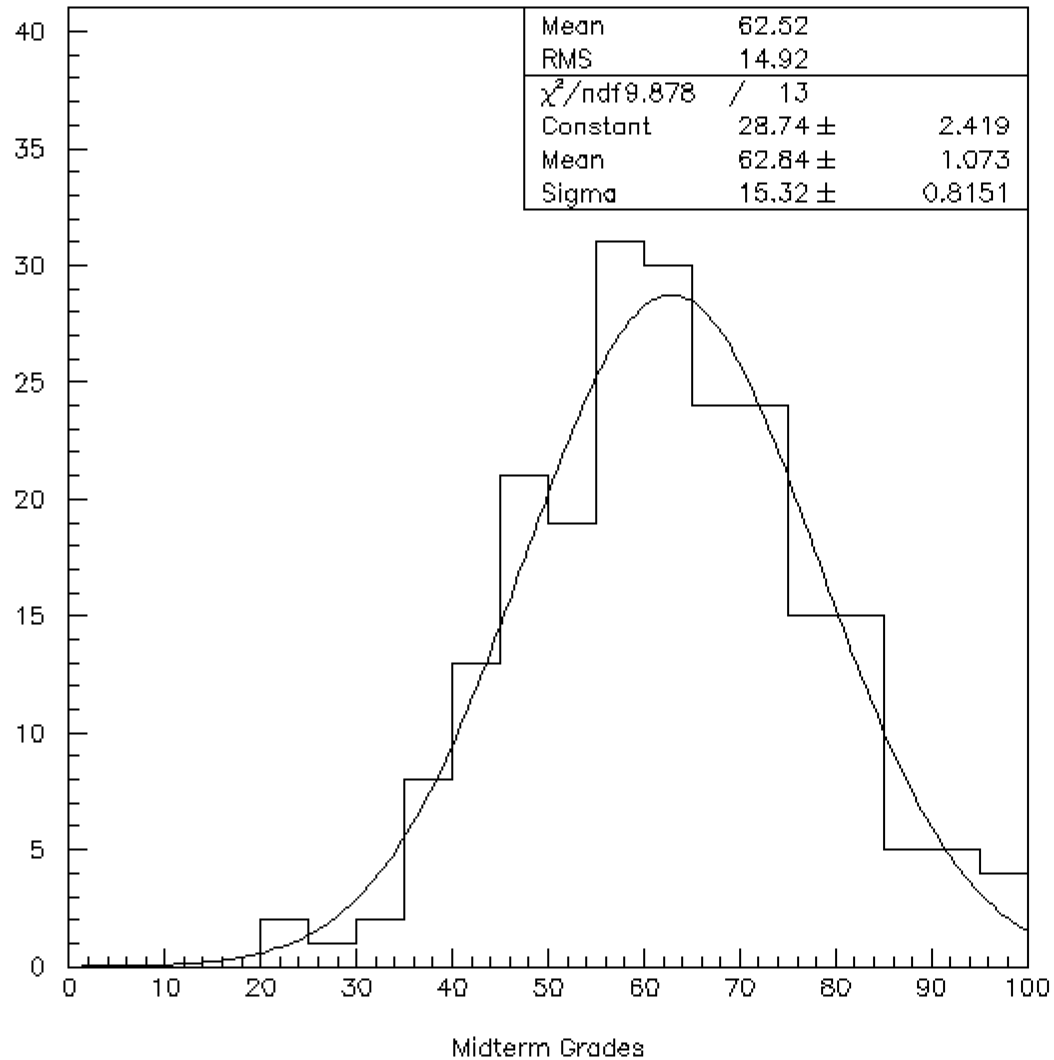
Fall 2004 Physics 3
Tu-Th Section

Claudio Campagnari

Lecture 11: 2 Nov. 2004

Web page:
http://hep.ucsb.edu/people/claudio/ph3-04/

Midterm Results



Pick up your graded exam from Debbie Ceder, Broida 5114
Solutions are posted at
http://hep.ucsb.edu/people/claudio/ph3-04/midterm.html

Last Lecture

- Electrical potential energy

$$W_{a\to b} = \int_a^b \vec{F} \cdot d\vec{l} = -[U(b) - U(a)]$$

Work done by electric force in moving a charge from a to b

Electric force

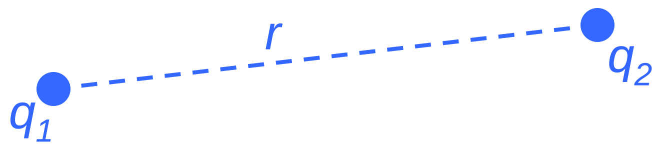
Potential energy at b

Potential energy at a

- Integral independent of path
  - Conservative force
- Potential energy defined up to arbitrary constant

Last Lecture (cont.)

- Potential energy of system of two charges



$$U(r) = k\frac{q_1 q_2}{r} + \text{Const.}$$

- Most often, take Const. = 0
- For many charges, sum over all pairs

$$U_{\text{total}} = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

- Conservation of energy

$$K_{\text{initial}} + U_{\text{initial}} = K_{\text{final}} + U_{\text{final}}$$

Last Lecture (cont.)

- Definition: if a charge q_0 in an electric field has electric potential energy U, then the electric potential is defined as

$$V = \frac{U}{q_0}$$

- Think of electric potential as "potential energy per unit charge"
- Much as electric field is "force per unit charge"
- Units: Joules/Couomb = Volt (V)

Last Lecture (cont.)

From definition V = U/q :

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \qquad \text{(potential due to a point charge)}$$

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

(potential due to a collection of point charges)

And also an (obvious) generalization:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

(potential due to a continuous distribution of charge)

Last Lecture (cont.)

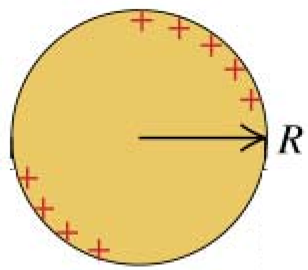
- Can also get V starting from electric field

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E \cos\phi \, dl$$

- This follows from definitions of V and U

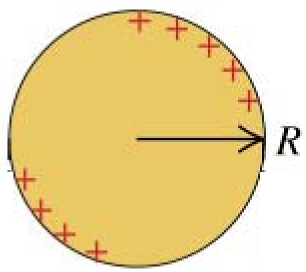
Example: potential from solid charged conducting sphere

- From the discussion of ~ two weeks ago based on Gauss' law <u>all the charge is on the surface</u>



- Divide the problem into two regions
  - ➢ Inside the sphere r < R
  - ➢ Outside the sphere
- Outside the sphere, we have seen already that E-field is same as if all charge was concentrated in the center

  → Potential same as for charge at the center, V=kQ/r



$\boxed{\text{Outside, } r>R \quad V(r) = kQ/r}$

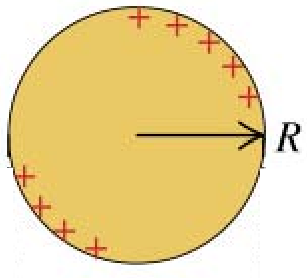
- Inside: from the discussion of ~ two weeks ago there is <u>no electric field inside a conductor</u>
  - Otherwise charges would be moving

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E \cos\phi \, dl$$

- If we take a and b to be anywhere inside

$$V_a - V_b = 0 \text{ or } V_a = V_b$$

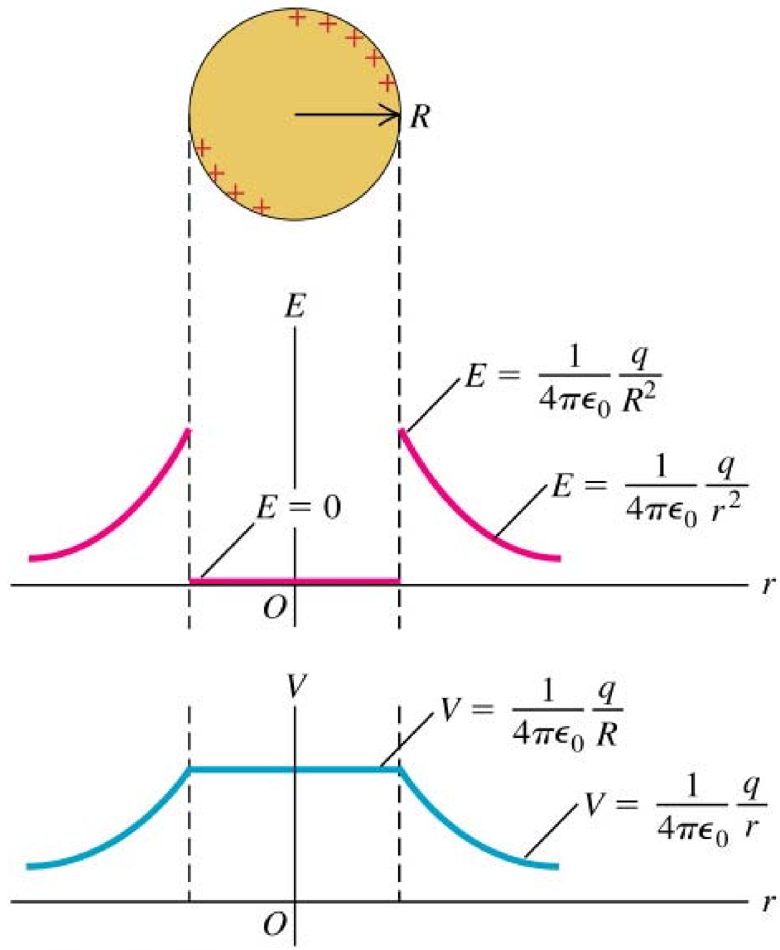
Potential is constant inside a conductor



Outside:	r>R	V(r) = kQ/r
Inside:	r<R	V(r) = constant

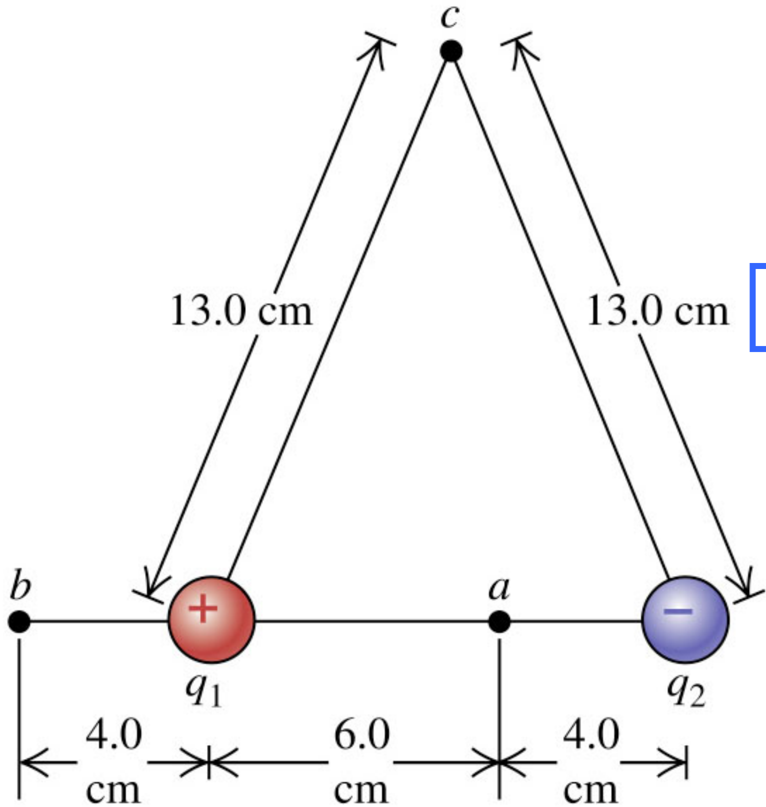
Q: What is the (constant) value of the potential inside?

A: By continuity, must be like on surface → V(r) = kQ/R



Another example: potential due to dipole

q_1 = + 12nC and q_2 = -12 nC, find potential at points a, b, c



$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

q = 12 nC   L=13 cm d_1=6 cm d_2=4 cm d_3 = 14 cm

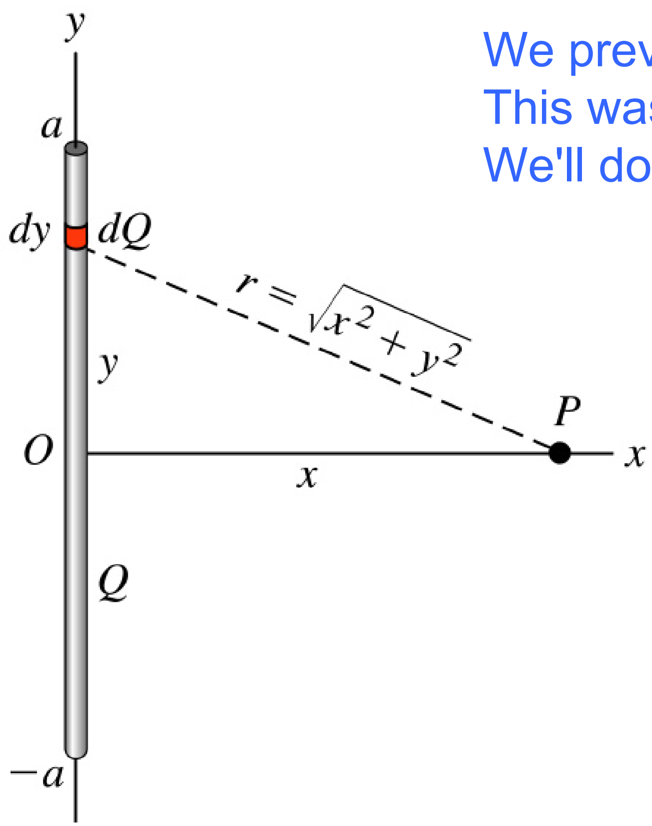
$$V_a = k\frac{q_1}{d_1} + k\frac{q_2}{d_2}$$
$$V_a = k\frac{q}{d_1} - k\frac{q}{d_2}$$
$$V_a = -900 \text{ V}$$

$$V_c = k\frac{q_1}{L} + k\frac{q_2}{L}$$
$$V_c = k\frac{q}{L} - k\frac{q}{L} = 0$$

$$V_b = k\frac{q_1}{d_2} + k\frac{q_2}{d_3}$$
$$V_b = k\frac{q}{d_1} - k\frac{q}{d_3}$$
$$V_b = 1930 \text{ V}$$

- Remember, we had calculated the electric field due to a dipole in a previous lecture
- The calculation of the electric potential is much easier!
- This is because electric field is a vector, so we have to worry about components, whereas electric potential is a scalar (a number with some units)

Another example: a line of charge



We previously calculated the electric field of this system.
This was done by adding up the fields due to small elements
We'll do the same thing, but for the potential!

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r}$$

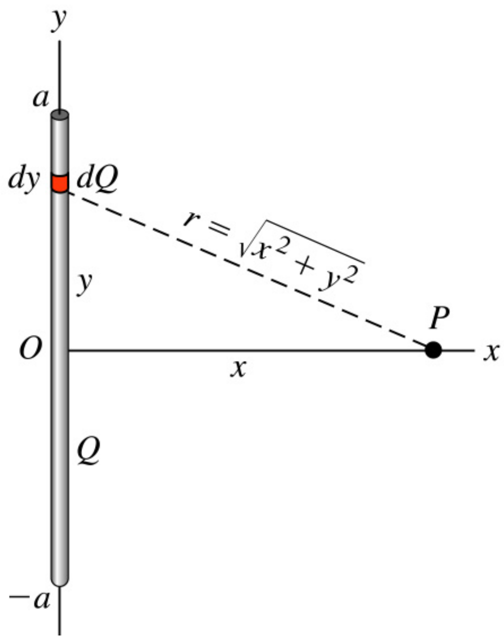
We'll be integrating over y
→need to express dq in terms of dy

dq = (Q/2a)dy

$$dV = k\frac{Q}{2a}\frac{dy}{\sqrt{x^2+y^2}}$$

$$V = k\frac{Q}{2a}\int_{-a}^{a}\frac{dy}{\sqrt{x^2+y^2}}$$

$$V = k\frac{Q}{2a}\ln\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}$$



$$V = k\frac{Q}{2a}\ln\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}$$

What happens for an infinite line charge?
Take the limit a → ∞
Linear charge density λ = Q/2a

$$V = k\lambda\ln\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}$$

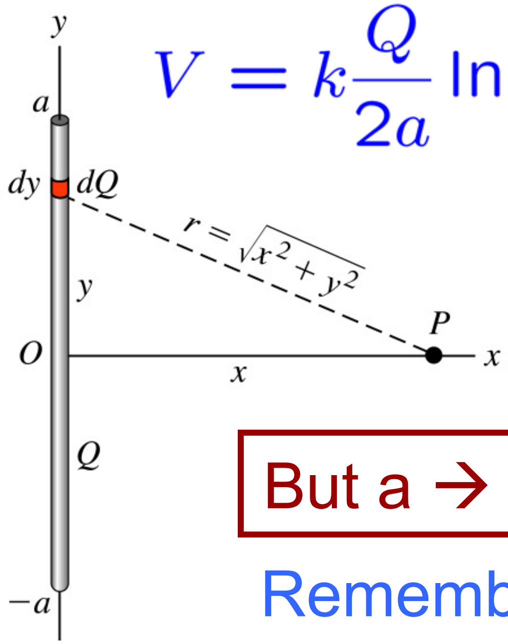
$$\sqrt{a^2+x^2}+a \to 2a$$

$$\sqrt{a^2+x^2}-a = a\sqrt{1+\frac{x^2}{a^2}}-a \to ?$$

Use binomial expansion

$$(1+y)^n = 1+ny+\frac{n(n-1)}{2!}y^2+\frac{n(n-1)(n-2)}{3!}y^3+\ldots$$

$$a\sqrt{1+\frac{x^2}{a^2}}-a \to a\left(1+\frac{1}{2}\frac{x^2}{a^2}\right)-a = \frac{1}{2}\frac{x^2}{a}$$



$$V = k\frac{Q}{2a}\ln\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a} \rightarrow k\lambda\ln\frac{2a}{\frac{1}{2}\frac{x^2}{a}} = k\lambda\ln\frac{4a^2}{x^2}$$

$$V = 2k\lambda\ln\frac{2a}{x}$$

But a → ∞.....so V also → ∞ .... What gives?

Remember:

Potential only defined up to additive constant

Only <u>differences</u> of potential meaningful

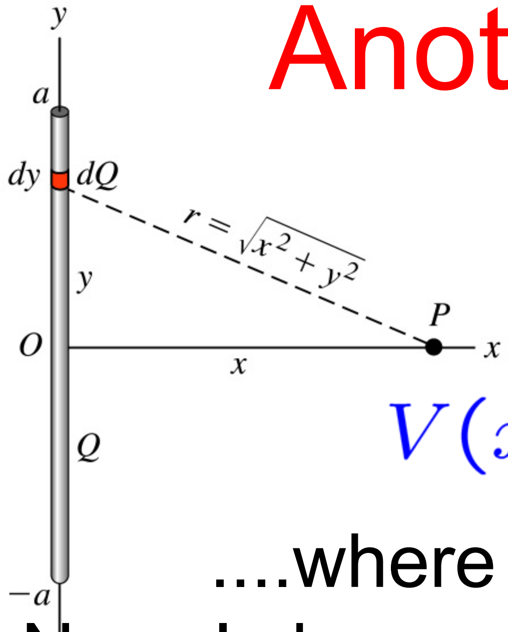
Consider <u>difference</u> of potential

$$V(x_1) - V(x_2) = 2k\lambda\ln\frac{2a}{x_1} - 2k\lambda\ln\frac{2a}{x_2}$$

$$V(x_1) - V(x_2) = 2k\lambda\ln\left(\frac{2a}{x_1}\frac{x_2}{2a}\right) = 2k\lambda\ln\frac{x_2}{x_1}$$

Finite and well-defined!

Another way to look at it....



$$V(x) = 2k\lambda \ln \frac{2a}{x} \quad (a \to \infty)$$

$$V(x) = 2k\lambda \ln(2a) + 2k\lambda \ln \frac{1}{x} + C$$

....where I added in an arbitrary constant.......

Now, I choose C such that at some value $x=x_0$, $V(x_0) = 0$

$$V(x_0) = 0 = 2k\lambda \ln(2a) + 2k\lambda \ln \frac{1}{x_0} + C$$

$$C = -2k\lambda \ln(2a) - 2k\lambda \ln \frac{1}{x_0}$$

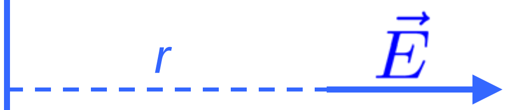
Plugging this value of C into the equation for V(x)....

$$V(x) = 2k\lambda \ln \frac{1}{x} - 2k\lambda \ln \frac{1}{x_0} = 2k\lambda \ln \frac{x_0}{x}$$

Finite and well-defined!

And yet another way to look at it...

- We saw in a previous lecture that the electric field due to infinite line of charge is



$$E(r) = 2k\frac{\lambda}{r}$$

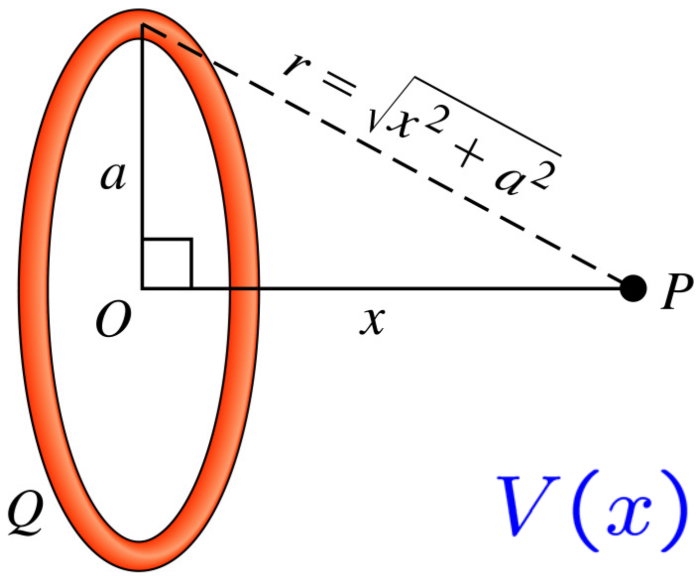
- We can calculate the potential <u>starting from E</u> (rather than from the charge distribution) using $V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}$

$$V(r_a) - V(r_b) = \int_{r_a}^{r_b} E dr = 2k\lambda \int_{r_a}^{r_b} \frac{dr}{r}$$

$$\boxed{V(r_a) - V(r_b) = 2k\lambda \ln\frac{r_b}{r_a}}$$

Same result as before!

Another example: ring of charge



First Method: use

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

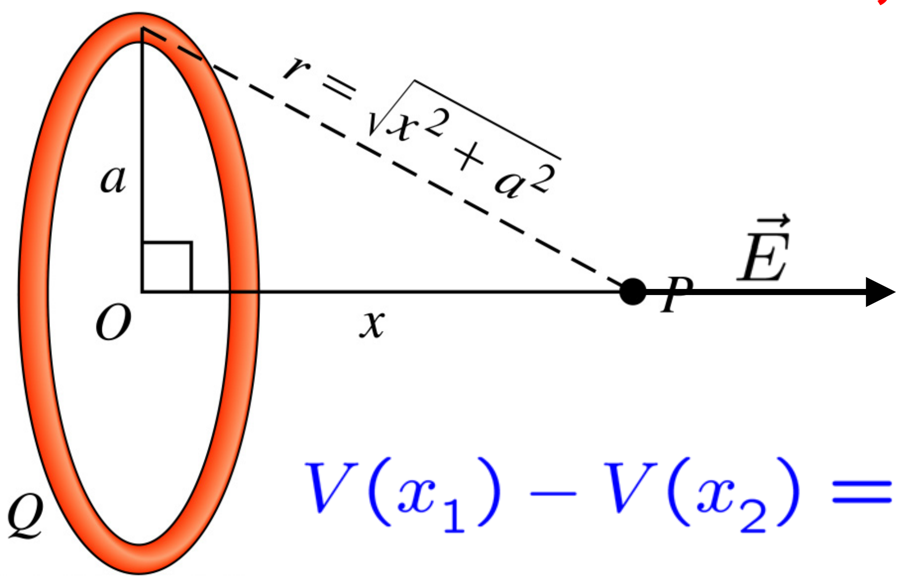
$$V(x) = k\int \frac{dq}{r}$$

We are integrating over the ring → r is constant, so we can pull it out of the integral sign:

$$V(x) = \frac{k}{r}\int dq = k\frac{Q}{r}$$

$$\boxed{V(x) = k\frac{Q}{\sqrt{x^2 + a^2}}}$$

Second Method, start from E-field



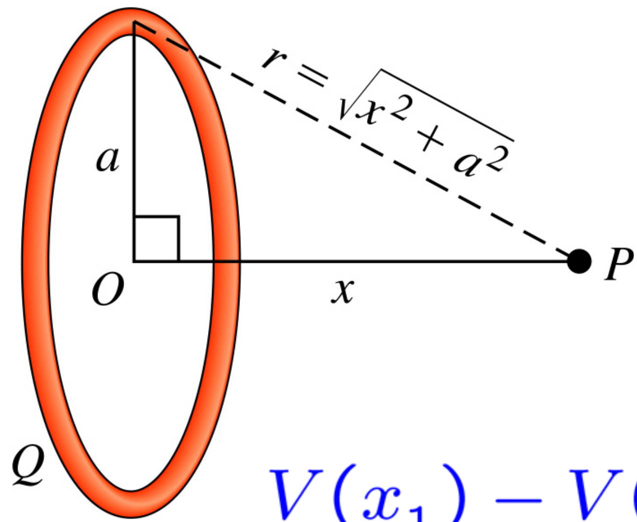
$$E = k\frac{Qx}{(x^2+a^2)^{\frac{3}{2}}}$$

(from lecture two weeks ago)

$$V(x_1) - V(x_2) = \int_{x_1}^{x_2} \vec{E} d\vec{x} = \int_{x_1}^{x_2} E dx$$

$$V(x_1) - V(x_2) = kQ \int_{x_1}^{x_2} \frac{x dx}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$V(x_1) - V(x_2) = kQ \left[\frac{-1}{\sqrt{x^2 + a^2}} \right]_{x_1}^{x_2}$$



$$V(x_1) - V(x_2) = kQ\left[\frac{-1}{\sqrt{x^2 + a^2}}\right]_{x_1}^{x_2}$$

$$V(x_1) - V(x_2) = kQ\left(\frac{1}{\sqrt{x_1^2 + a^2}} - \frac{1}{\sqrt{x_2^2 + a^2}}\right)$$

Take $x_2 = \infty$ and $V(\infty)=0$

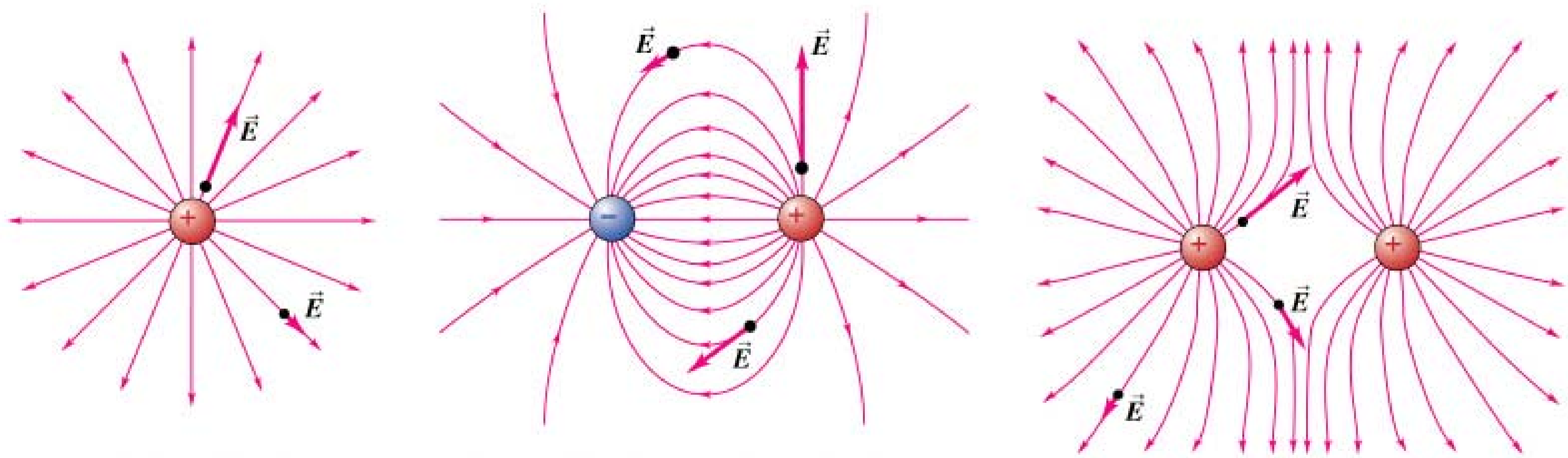
Choosing $V(\infty)=0$ is equivalent to choosing the arbitrary constant

$$V(x) - V(\infty) = \boxed{V(x) = kQ\frac{1}{\sqrt{x^2 + a^2}}}$$

Same result as before!

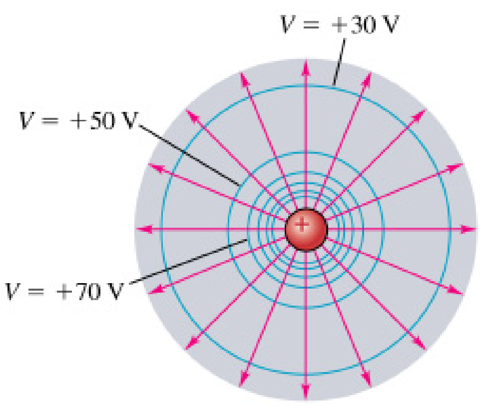
Equipotential lines and surfaces

- Remember electric field lines
  - ➢ A tool to visualize the electric field
  - ➢ Lines drawn parallel to the electric field
  - ➢ With arrows pointing in the direction of the electric field

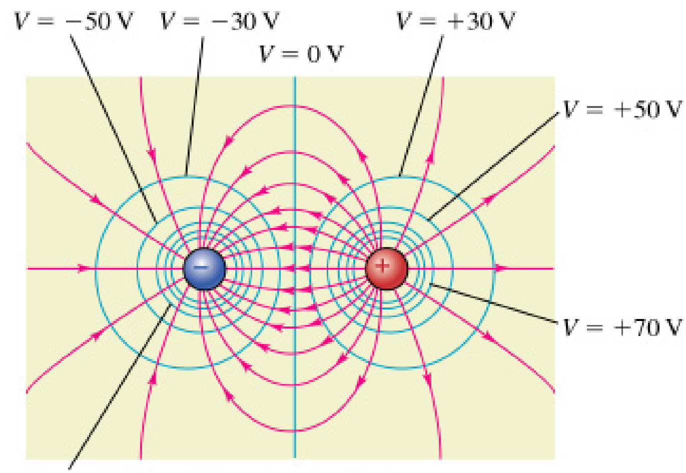


Equipotentials

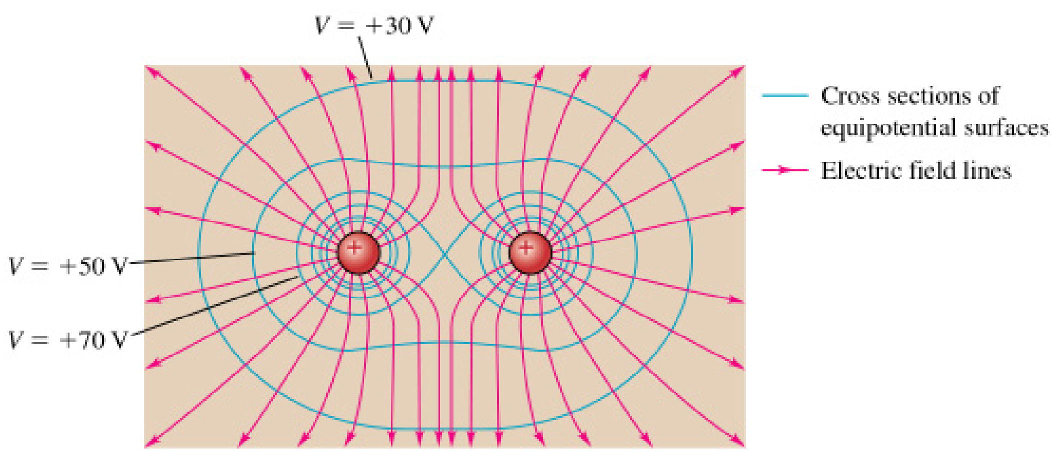
Lines (or surfaces in 3D) of constant potential



(a) A single positive charge



(b) An electric dipole



(c) Two equal positive charges

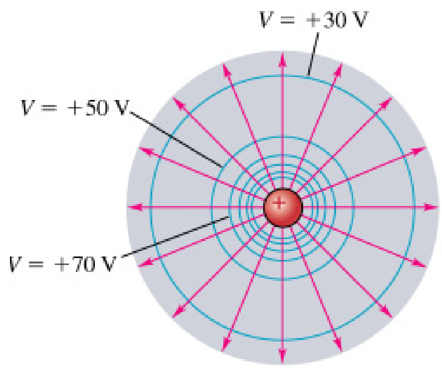
Equipotentials, continued

- Note, from previous slides, that equipotentials are perpendicular to field lines
- Why?
- Potential does not change as we move along an equipotential

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} \quad = 0 \text{ along equipotential}$$

$$\rightarrow \quad \vec{E} \text{ perpendicular to } d\vec{l}$$

where $d\vec{l}$ is along equipotential



(a) A single positive charge

More on equipotentials

- It is customary to draw equipotentials at constant intervals in ΔV
  - ➢ The figure above has V = 30, 50, 70... V, i.e., ΔV = 20 V
- Then, the field is stronger where the density of equipotentials is highest
  - ➢ Since ΔV is the same between each pair of equipotentials, this way the work done in moving a test charge between pairs of equipotentials is always the same. Since work = Force x Distance, and Force = charge x E-field, when the distance is smallest, the E-field must be highest

Equipotentials and Conductors

- We argued previously that the electric field must be perpendicular to the surface of a conductor
- This is because otherwise the charges on the surface of the conductor would be moving
- It then follows that the surface of a conductor is an equipotential

Potential Gradient

- We will now derive a fundamental relationship between potential and electric field

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$$

- But

$$V_a - V_b = \int_b^a dV = -\int_a^b dV$$

- This must hold for <u>any</u> (a,b) pair and <u>any</u> path between the two
- For this to be true then

$$-dV = \vec{E} \cdot d\vec{l}$$

$$-dV = \vec{E} \cdot d\vec{l}$$

- Write out the components

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$
$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

- Then, in terms of components:

$$-dV = E_x dx + E_y dy + E_z dz$$

- Suppose the displacement is in the x-direction
- Then dy=dz=0, and $-dV=E_x dx$, or

$$E_x = -\left(\frac{dV}{dx}\right)_{\text{(constant y,z)}}$$

$$E_x = -\frac{\partial V}{\partial x}$$

- Can do same thing for the other two components

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

(components of $\vec{E}$ in terms of V)

$$\vec{E} = -\left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \qquad (\vec{E} \text{ in terms of } V)$$

- Or in short-hand notation

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

- $\vec{\nabla}$ is called the "gradient" or the "grad"

$$\boxed{\vec{E} = -\vec{\nabla}V = -\left(\hat{i}\frac{\partial V}{dx} + \hat{j}\frac{\partial V}{dy} + \hat{k}\frac{\partial V}{dz}\right)}$$

- If we shift V → V + Const. the E-field does not change
- Makes sense, since V is only defined up-to arbitrary constant
- The expression above for the gradient is in "Cartesian Coordinates"
  - Cartesian coordinates: x,y,z
- One important result:
  - If V is a function of r (and not of angle) then

$$E_r = -\frac{\partial V}{\partial r}$$

Simple applications of gradient law

- Point charge:

$$V(r) = k\frac{q}{r}$$

$$E(r) = -\frac{\partial V}{\partial r} = k\frac{q}{r^2}$$

- Infinite line of charge

$$V(r) = 2k\lambda \ln \frac{r_0}{r}$$

Slide 16, this lecture

$$V(r) = \underbrace{2k\lambda \ln(r_0)}_{\text{constant}} + 2k\lambda \ln \frac{1}{r}$$

$$E(r) = -\frac{\partial V}{\partial r} = -2k\lambda \frac{\partial\left(\ln \frac{1}{r}\right)}{\partial r} = +2k\frac{\lambda}{r}$$