Fall Quarter 2023 – UCSB Physics 29  
Homework 6  
Due Wednesday November 22, 0:00 am

Read carefully the instructions on the website on how to prepare your homework for turning it in to the TA.

To do these problems, refer to the slides and exercises of weeks 6 and 7.

• Problem 1
  Let’s calculate the period of oscillation of a pendulum without resorting to the small angle approximation. Let $L$ be the length of the pendulum and $M$ the mass, and take the zero of potential energy at the equilibrium position. If the angle between the pendulum and the vertical is $\theta$, the potential energy is $U(\theta) = MgL(1 - \cos \theta)$. If the maximum angle of oscillation is $\alpha$, conservation of energy gives:

$$\frac{1}{2}Mv^2 + MgL(1 - \cos \theta) = MgL(1 - \cos \alpha)$$

Since $v$ is tangential, $v = L\frac{d\theta}{dt}$, so we get

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}(\cos \theta - \cos \alpha)}$$

Using the identity $\cos A = 1 - 2\sin^2 \frac{A}{2}$ this can be written as

$$\frac{d\theta}{dt} = \sqrt{\frac{4g}{L}\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}\right)}$$

The time taken to go from $\theta = 0$ to $\theta = \alpha$ is one fourth of the period $T$. Therefore

$$\frac{T}{4} = \sqrt{\frac{L}{4g}} \int_0^\alpha \frac{d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}}$$

At this point we could integrate numerically. It is a bit messy since the integrand goes to $\infty$ at $\theta \to \alpha$. Instead we can substitute $\sin \beta = \frac{\sin \theta/2}{\sin \alpha/2}$ which after some algebra gives

$$T = \frac{2}{\pi}T_0 \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}}$$

with $k = \sin \frac{\alpha}{2}$ and $T_0 = 2\pi\sqrt{L/g}$ is the period of the pendulum in the small angle approximation. This integral is an elliptic integral of the first kind. Integrate numerically this equation, using whatever method you like, to plot $T/T_0$ as a function of $\alpha$. 
Plot $\alpha$ in degrees between $1^\circ$ and $90^\circ$.

To check yourself, you might want to compare with the Python function that calculates this integral for you, see

https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.ellipk.html

- **Problem 2**

A particle physics experiment is searching for dark matter by looking for a signal of dark matter particles bumping into nuclei and causing nuclei to recoil. They have a detector that is able to "see" the signature of a nuclear recoil. They have also gone to great pains to limit sources of background that may cause spurious signals, e.g., cosmic rays, radioactivity in or near the detector, etc.

They take data for a whole year, but despite their best efforts they are not able to eliminate all backgrounds. Through all sorts of difficult studies they conclude that they should expect on average to see $\mu_B = 1.5$ background events that mimic the dark matter signature. They see $N = 2$ candidate dark matter events.

Obviously, they do not have any evidence for dark matter interactions in their detector, since for a mean of 1.5 background events, the 2 events that they have could easily be all background. However, one can still extract information from such an experiment.

Clearly their result is incompatible with Nature being such that on average $\mu_S = 1000$ dark matter events would have been seen, since they only see $N = 2$, and the average background level was only $\mu_B = 1.5$. On the other hand, the experiment would have been insensitive to a scenario where $\mu_S = 0.01$. So what they need to do is to report a "limit" on $\mu_S$, i.e., the largest possible value of $\mu_S$ that the experiment is statistically sensitive too.

Conventionally, what is done is to find the highest possible value of $\mu_S$ such that the experiment would have seen $\leq N$ events (i.e., 0, 1, or 2) with > 5% probability. This is the so-called "95% confidence level upper limit on $\mu_S$". Technically this method to quote a 95% limit is called "Classic Frequentist". There are several other statistical analysis methods on the market.

In order to calculate the limit, find the $\mu \equiv \mu_B + \mu_S$ such that $p(0|\mu) + p(1|\mu) + p(2|\mu) = 0.05$, where $p(n|\mu)$ is the Poisson probability of having $n$ counts for a mean $\mu$. Since $\mu_B$ is given with only one digit after the decimal point, there is no need to be more precise than that when quoting $\mu_S$.

A real statistical analysis would be more complex because it would have to account for an uncertainty in $\mu_B$ (nothing is ever perfectly known in physics). Eventually, the results must be reported in some units that other people can understand. So for example $\mu_S$ must be scaled by the fact that in this particular experiment the efficiency for seeing a nuclear recoil is not 100%, and the efficiency is not perfectly known either, etc. etc.
• Problem 3

Take the first 2,000 (out of 50,000) entries in the data set from Homework 4, Problem 3. Plot the data in a histogram with 50 bins between $x = 0$ and $x = 25$, in both linear and log scale. You can steal as much as you want from the posted solutions to that problem. Also, display Poisson error bars $\pm \sqrt{N}$, where $N$ is the bin content, on each bin.

**Hint:** Remember that the `ax.hist` function returns a tuple whose elements are two numpy arrays. The first array is the contents of the bins, the second array is the bin edges.

Now fit the first few bins to $y = f(x) = Ae^{-Bx}$ but limit your fit to the region at low $x$ where the Poisson distribution is reasonably close to the Gaussian. This is a judgement call, refer to the discussion in the Week 7 lecture.

Do the fit in two ways:

1. Using `curve_fit` to fit $y = f(x) = Ae^{-Bx}$
2. Using `polyfit` to fit $\log y = \log A - Bx$

and compare the results for $B$. Think carefully about what uncertainties you should use in the 2nd case.

You will probably not get identical results. Think about why that might be (the thinking part is not for credit).