Probability Distribution Functions (PDFs) and Probability Mass Functions (PMFs)

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Suppose that we wanted to characterize the probability of some quantity to have some specific value. We need to make a distinction between continuous vs discrete variables.

1 Continuous variables: PDF

For example, suppose we are interested in the probability that a UCSB student is 6 ft tall. A person's height is a continuous variable which can take any real numerical value between 0 (I guess...theoretically...) and $+\infty$. Since there are an infinite possible values for the height variable, the probability that the height is **exactly** 6 ft is zero.

What we can talk about is the probability that a student's height is somewhere between 6 ft and 6 ft 1 inch. Therefore, for a continuous variable x we can define a function $f(x)$ such that the probability of the variable taking a value between x and $x + dx$ is $f(x)dx$. The function $f(x)$ is called Probability Distribution Function (PDF).

Since the probability of taking *some* value is 1, the PDF must be normalized to reflect this basic fact, ie:

$$\int_{x_{\min}}^{x_{\max}} f(x)dx = 1$$

where $x_{\min}$ and $x_{\max}$ are the minimum and maximum value that the variable x can possibly have.

It is often convenient to define the Cumulative Distribution Function (CDF) as

$$F(x) = \int_{x_{\min}}^{x} f(x')dx'$$

Then $F(x)$ is the probability that the variable of interest take a value $< x$. Conversely, $1 - F(x)$ is the probability for the variable to take a value $> x$. By definition $f(x) = dF(x)/dx$.

1.1 Change of variables

Suppose we want to change variables from x with PDF $f(x)$ to $y = y(x)$. What is the PDF $g(y)$ for y?

For simplicity, we only consider a monotonic transformation $x \to y(x)$, ie, we do not allow the possibility of $y(x_i) = y(x_j)$ unless $x_i = x_j$. For now we also require $dy/dx > 0$. This positive sign requirement for the derivative will be lifted at the end.

Let $F(x)$ be the CDF of x, and $G(x)$ be the CDF of y.

$$G(y) = F(x(y))$$

Taking derivative with respect to y of both sides:

$$\frac{dG(y)}{dy} = g(y) = \frac{d}{dy}F(x(y)) = \frac{dF(x(y))}{dx}\frac{dx}{dy}$$

Thus

$$\boxed{g(y) = f(x(y))\left|\frac{dx}{dy}\right|}$$

where the absolute value takes care of the case $dy/dx < 0$, in which case the equation for $G(y)$ would have been written as $G(y) = 1 - F(x(y))$.

2 Discrete variables: PMF

For discrete variables, eg, the number of siblings of UCSB students (0, 1, 2, 3, ...) the probability of a specific (integer, in this case) number of siblings is some finite value. Then we can define a function called the Probabiity Mass Function (PMF) $f(x)$ which is simply the probability that the variable of interest takes the specific value x. The PMF must also be normalized such that

$$\sum f(x_i) = 1$$

where the sum runs over all possible values of x. One can also define a Cumulative Mass Function (CMF) $F(x)$ in the obvious way

$$F(x) = \sum_{x_i \leq x} f(x_i)$$