Spring Quarter 2026 – UCSB Physics 25
Midterm Solutions

Problem 1

A negatively-charged pion (π^-, rest mass m_π) decays at rest into an electron (rest mass m_e) and a neutrino (rest mass $m_\nu = 0$). What is the magnitude of the 3-momentum of the electron? Of the neutrino? (You can do this problem using $c = 1$ if you wish).

Solution

Since the pion decays *at rest*, conservation of 3-momentum requires the electron and neutrino to be emitted back-to-back with equal and opposite momenta. Their magnitudes are therefore equal:

$$|\vec{p}_e| \;=\; |\vec{p}_\nu| \;\equiv\; p. \tag{1}$$

Energy conservation. The initial energy is the pion rest energy, $E_i = m_\pi$. The final-state energies are

$$E_e \;=\; \sqrt{p^2 + m_e^2}, \qquad E_\nu \;=\; p \quad (\text{since } m_\nu = 0). \tag{2}$$

Energy conservation gives

$$m_\pi \;=\; \sqrt{p^2 + m_e^2} \;+\; p. \tag{3}$$

Solving for p. Isolate the square root and square both sides of (3):

$$\sqrt{p^2 + m_e^2} \;=\; m_\pi - p, \tag{4}$$

$$p^2 + m_e^2 \;=\; m_\pi^2 - 2\,m_\pi\,p + p^2, \tag{5}$$

$$2\,m_\pi\,p \;=\; m_\pi^2 - m_e^2. \tag{6}$$

Therefore

$$\boxed{p \;=\; |\vec{p}_e| \;=\; |\vec{p}_\nu| \;=\; \frac{m_\pi^2 - m_e^2}{2\,m_\pi}} \tag{7}$$

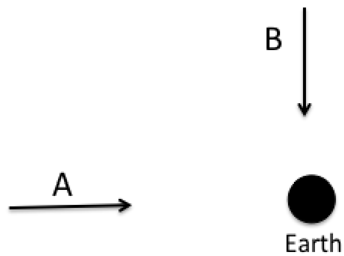
Restoring factors of c. Putting the speed of light back in,

$$\boxed{p \;=\; \frac{(m_\pi^2 - m_e^2)\,c}{2\,m_\pi}} \tag{8}$$

Check (massless limit). If $m_e \to 0$, the result reduces to $p = m_\pi/2$, exactly what one expects for a two-body decay into two massless particles—each carries half of the parent's rest energy.

Problem 2

Spaceships A and B are approaching the earth as shown below at speeds of $0.9c$ as measured by an observer on the earth. What is the speed of spaceship B as measured by the pilot of spaceship A. (Reminder: *speed* is the magnitude of velocity).



Setup of the solution

Let S be the Earth's frame, with $\hat{x}$ pointing to the right (the direction of A's motion) and $\hat{y}$ pointing upward. Then

$$\vec{u}_A = (+0.9c,\ 0), \qquad \vec{u}_B = (0,\ -0.9c). \tag{9}$$

Let S' be A's rest frame. S' moves with velocity $v = +0.9c$ along $\hat{x}$ relative to S, so we transform B's velocity into S' using the relativistic velocity-addition formulas for a boost along $\hat{x}$:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}, \tag{10}$$

$$u'_y = \frac{u_y}{\gamma\,(1 - u_x v/c^2)}, \tag{11}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Apply to B

For B we have $u_x = 0$ and $u_y = -0.9c$, so the denominator $1 - u_x v/c^2 = 1$ and

$$u'_{B,x} = \frac{0 - 0.9c}{1} = -0.9c, \tag{12}$$

$$u'_{B,y} = \frac{-0.9c}{\gamma} = -0.9c\,\sqrt{1 - v^2/c^2}. \tag{13}$$

With $v = 0.9c$,

$$\gamma = \frac{1}{\sqrt{1 - 0.81}} = \frac{1}{\sqrt{0.19}}, \qquad \frac{1}{\gamma} = \sqrt{0.19}, \tag{14}$$

so

$$\vec{u}'_B \;=\; \big(-0.9c,\;\; -0.9c\,\sqrt{0.19}\big). \tag{15}$$

Speed (magnitude)

$$
\begin{aligned}
|\vec{u}'_B|^2 &= (0.9c)^2 + (0.9c)^2\,(0.19) && (16)\\
&= (0.9c)^2\,(1 + 0.19) && (17)\\
&= (0.9)^2\,(1.19)\,c^2 && (18)\\
&= 0.9639\,c^2. && (19)
\end{aligned}
$$

$$
\boxed{|\vec{u}'_B| \;=\; 0.9c\,\sqrt{1.19} \;=\; \frac{9\sqrt{119}}{100}\,c \;\approx\; 0.982\,c} \tag{20}
$$

Remarks

- The x-component of $\vec{u}'_B$ is $-0.9c$: since B has no x-motion in S, in A's frame it simply streams backward at exactly A's Earth-frame speed.
- The transverse (y) component is reduced by a factor $1/\gamma$ compared with its value in S. This is the standard "transverse-velocity" effect: the Earth-frame clocks used to measure u_y run slow as seen by A, so A sees B's downward motion advance more slowly than an Earth observer does.
- The result $0.982c < c$ is, of course, less than the speed of light, as required.

Problem 3

A futuristic train moving in straight line with a uniform speed of $0.8c$ passes a series of communication towers. The spacing of the towers according to an observer on the ground is 3.0 km. A passenger on the train uses an accurate stopwatch to measure how often the train passes a tower.
(a) What is the time interval the passenger measures between the passing of one tower and the next? Use $c = 3.0 \cdot 10^8$ m/sec.
(b) What is this time as measured by the observer on the ground?

Lorentz factor

$$\gamma \;=\; \frac{1}{\sqrt{1 - v^2/c^2}} \;=\; \frac{1}{\sqrt{1 - 0.64}} \;=\; \frac{1}{\sqrt{0.36}} \;=\; \frac{1}{0.6} \;=\; \frac{5}{3}. \tag{21}$$

(a) Time measured by the passenger

The two events "passenger meets tower n" and "passenger meets tower $n+1$" both occur at the passenger's own location in the train frame, so the passenger's stopwatch measures the *proper time* $\Delta\tau$ between them.

In the train's frame the tower spacing is length-contracted,

$$L \;=\; \frac{L_0}{\gamma} \;=\; (3000 \text{ m})(0.6) \;=\; 1800 \text{ m}, \tag{22}$$

and the towers stream past at speed $v = 0.8c$. Hence

$$\Delta\tau \;=\; \frac{L}{v} \;=\; \frac{1800 \text{ m}}{0.8\,(3.0 \times 10^8 \text{ m/s})} \tag{23}$$

$$\;=\; \frac{1800}{2.4 \times 10^8} \text{ s} \;=\; 7.5 \times 10^{-6} \text{ s}. \tag{24}$$

$$\boxed{\Delta\tau \;=\; 7.5 \;\mu\text{s}} \tag{25}$$

(b) Time measured by the ground observer

In the ground frame the towers are $L_0 = 3000$ m apart and the train moves at $v = 0.8c$, so

$$\Delta t \;=\; \frac{L_0}{v} \;=\; \frac{3000 \text{ m}}{2.4 \times 10^8 \text{ m/s}} \tag{26}$$

$$\;=\; 1.25 \times 10^{-5} \text{ s}. \tag{27}$$

$$\boxed{\Delta t \;=\; 12.5 \;\mu\text{s}} \tag{28}$$

Consistency check

The two times should be related by time dilation, $\Delta t = \gamma\,\Delta\tau$, since $\Delta\tau$ is a proper time:

$$\gamma\,\Delta\tau \;=\; \frac{5}{3}\,(7.5 \;\mu\text{s}) \;=\; 12.5 \;\mu\text{s} \;=\; \Delta t. \quad \checkmark \tag{29}$$

Problem 4

A neutral pion (π^0) of mass M and momentum P is traveling in the positive x-direction. It decays into two massless photons. Photon 1 has momentum k_1 and is traveling in the positive x-direction; photon 2 has momentum k_2 and is traveling in the negative x-direction. Find k_1 and k_2 in terms of M and P. (You can do this problem using $c = 1$ if you wish).
Note: k_1 and k_2 are **magnitudes** of momenta, and therefore should be taken as **positive** quantities.

Setup

In units with $c = 1$, the pion's energy is

$$E_\pi = \sqrt{P^2 + M^2}. \tag{30}$$

For a massless photon, $E = |\vec{p}|$, so the two photons have energies k_1 and k_2 respectively, with x-momenta $+k_1$ and $-k_2$.

Conservation laws

Momentum (x-component).

$$P = k_1 - k_2. \tag{31}$$

Energy.

$$\sqrt{P^2 + M^2} = k_1 + k_2. \tag{32}$$

Solve

Adding and subtracting Eqs. (31) and (32),

$$2\,k_1 = \sqrt{P^2 + M^2} + P, \tag{33}$$

$$2\,k_2 = \sqrt{P^2 + M^2} - P. \tag{34}$$

Therefore

$$\boxed{k_1 = \tfrac{1}{2}\Big(\sqrt{P^2 + M^2} + P\Big), \qquad k_2 = \tfrac{1}{2}\Big(\sqrt{P^2 + M^2} - P\Big).} \tag{35}$$

Or, restoring the factors of c:

$$\boxed{k_1 = \tfrac{1}{2}\Big(\sqrt{P^2 + M^2c^2} + P\Big), \qquad k_2 = \tfrac{1}{2}\Big(\sqrt{P^2 + M^2c^2} - P\Big).} \tag{36}$$

Checks

- Both k_1 and k_2 are positive, since $\sqrt{P^2 + M^2} \geq |P|$.
- *Pion at rest* ($P = 0$): $k_1 = k_2 = M/2$. Each photon carries half the rest energy and they fly off back-to-back, as expected.
- *Invariant-mass check.* The two photons must have invariant mass M. For collinear photons going in opposite directions,

  $$M^2 = (k_1 + k_2)^2 - (k_1 - k_2)^2 = 4\,k_1 k_2, \tag{37}$$

  and indeed

  $$4\,k_1 k_2 = \big(\sqrt{P^2 + M^2}\big)^2 - P^2 = M^2. \quad \checkmark \tag{38}$$