Four Vectors, Part 1

Claudio Campagnari

9 April 2026

1 Lorentz Transformation of time and space

Consider a frame S with coordinates x, y, z, t and a frame S' with coordinates x', y, z', t' moving with velocity v in the positive x-direction with respect to S. The coordinate transformation between the S and S' coordinates is given by the Lorentz transformation equations:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned} \tag{1}$$

Where $\gamma = 1/\sqrt{1 - v^2/c^2}$. This type of transformation is also called "boost".

In classical physics the transformation (Galilean transformation) is simply

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{2}$$

In the limit $v << c$ the Lorentz transformation reduces to the Galilean transformation, as it should. These equations can be generalized for arbitrary directions of the relative velocity of the two frames.

In three dimensional (Euclidean) space, let $\vec{r}$ be the position vector with components (x, y, z). The length, or more conveniently, the square of the length of the vector $\vec{r}$ is the same (ie: **invariant**) under rotation of the axes, in other words $r^2 = x^2 + y^2 + z^2$ is "an invariant".

The length of the vector is also invariant under a Galilean transformation. careful though: If you look at equation 2 you might be confused, since $r^2 = x^2 + y^2 + z^2 \neq r'^2 = x'^2 + y'^2 + z'^2$. The point is that $\vec{r}$ is defined with respect to the origin of S, and $\vec{r'}$ (as written) is defined with respect to the origin of S' and the origin of S does not transform into the origin of S'. If you instead were to calculate the square of the distance in S' between the the point at coordinates (x', y', z') and the coordinates of the origin of S in the S' coordinates after a Galilean transformation, ie, $(-vt, 0, 0)$ you would recover the invariant $x^2 + y^2 + z^2$.

A coordinate rotation in 3D space by an angle θ around the z axis is written as

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= \ z \end{aligned} \tag{3}$$

The length of the position vector is invariant under this coordinate transformation, ie, $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$.

In homework 1 you saw that the Lorentz boost transformation can be written as a rotation of axes using hyperbolic sines and cosines:

$$\begin{aligned} x' &= x\cosh\alpha - ct\sinh\alpha \\ t' &= t\cosh\alpha - \frac{x}{c}\sinh\alpha \\ y' &= y \\ z' &= z \end{aligned} \tag{4}$$

where $\tanh\alpha = \frac{v}{c}$

Equations 1 and 4 suggests that we should now work in a four-dimensional space, where the fourth dimension is time t, or ct if we want to preserve the same dimensions for all the components. This four-dimensional space is called Minkowski space. A vector in this space is called a **four-vector**. A vector in Euclidean space is then called a **three-vector**. The way to refer to the fourth coordinate is convention dependent, see below for the various notation conventions:

1: $x_1, x_2, x_3 = x, y, z$ and $x_4 = ct$
2: $x_0 = ct$ and $x_1, x_2, x_3 = x, y, z$
3: $x_1, x_2, x_3 = x, y, z$ and $x_4 = ict$

The 3rd convention is old and, I dare say, obsolete. The 2nd convention is the most widely used. The book by K&K uses the first convention.

Just like the length of a three-vector is invariant under spacial rotation, one can define a quantity that is invariant under a Lorentz boost. This quantity is called the **square of the length of the four vector** or just **the square of the 4-vector**, and is given by

$$l^2 = \pm(x^2 + y^2 + z^2 - c^2t^2) = \pm(r^2 - c^2t^2) \tag{5}$$

where r is the length of the 3-vector, and, the $\pm$ is, once again, convention dependent. This convention is associated to the choice of "metric", $(+,-,-,-)$ or $(-,+,+,+)$.....don't worry about this for now, just remember that when people talk about "the metric" in Special Relativity they mean the choice of sign convention.

Note the relative minus sign between the space coordinates and the time coordinate in the definition of the square of the four-vector. In Minkowski space one of the coordinates is "special". Incidentally, the reason to introduce the factor of $i = \sqrt{-1}$ in front of the time coordinate in the 3rd notation convention listed above is to "hide" the minus sign in the definition of square of the four-vector which would now be simply $l^2 = \sum x_i^2$. Just conventions...