Before  After

\[ W \quad M \rightarrow p \quad p' \]

Initial Momentum \( W \)
Final Momentum \( p \) \Rightarrow \( p = W \)
Initial Energy \( W = W + M \)
Final Energy \( \sqrt{p'^2 + p^2} \)

Conservation of Energy

\[ W^2 = (W + M)^2 = \sqrt{p'^2 + p^2} = \sqrt{p'^2 + W^2} \]

Square both sides

\[ W^2 + M^2 + 2MW = M'^2 + W^2 \]

\[ M' = \sqrt{M^2 + 2MW} \]

Since \( W \) has units of energy, restore factors of \( c \) to get units right

\[ M' = \sqrt{M^2 + \frac{2MW}{c^2}} \]
2. From the point of view of earth, two events

\[ \begin{align*}
L'' \rightarrow 2\nu \\
L' \rightarrow \nu \\
x = 0 \\
t = 0 \\
\end{align*} \]

\[ \begin{align*}
L'' \rightarrow 2\nu \\
L' \rightarrow \nu \\
x = ? \\
t = ? \\
\beta = \frac{\nu}{c} \\
\end{align*} \]

Length contraction:
\[ L' = \sqrt{1-\beta^2} L \]
\[ L'' = \sqrt{1-4\beta^2} L \]

Eqn of motion for front of slow ship:
\[ x = L' + \nu t \]
Eqn of motion for back of fast ship:
\[ x = -L'' + 2\nu t \]

At the coordinates \((x, t)\) of the second event:
\[ L' + \nu t = -L'' + 2\nu t \implies t = \frac{L' + L''}{2\nu} = \frac{\sqrt{1-\beta^2} + \sqrt{1-4\beta^2}}{\sqrt{1-\beta^2}} \frac{L}{\nu} \]

And also:
\[ x = L' + \nu t = \left[2\sqrt{1-\beta^2} + \sqrt{1-4\beta^2}\right] L \]

(b) Two ways of doing this - First way: coordinate transform

Let's go to the rest frame of the slow ship

\[ t' = \frac{t - \frac{\nu x}{c^2}}{\sqrt{1-\frac{\nu^2}{c^2}}} \quad \text{with} \quad x = \frac{1}{\sqrt{1-\beta^2}} \]

Substitute \((x, t)\) from part (a)

\[ t' = \frac{L}{\nu} + \frac{\sqrt{1-4\beta^2}}{\sqrt{1-\beta^2}} \sqrt{\nu} - \frac{2\nu - L}{c^2} - \frac{x \cdot L}{c^2} \frac{\sqrt{1-4\beta^2}}{\sqrt{1-\beta^2}} \]

Note: \(\frac{\nu t}{c^2} = \beta \frac{L}{\nu} \)

\[ t' = \frac{L}{\nu} \left[1 - 2\beta^2 + \frac{\sqrt{1-4\beta^2}}{\sqrt{1-\beta^2}} - \frac{\beta^2 \sqrt{1-4\beta^2}}{\sqrt{1-\beta^2}}\right] \]

\[ t' = \frac{L}{\nu} \left[1 - 2\beta^2 + (1-\beta^2) \frac{\sqrt{1-4\beta^2}}{1-\beta^2}\right] \]

\[ t' = \frac{L}{\nu} \left[1 - 2\beta^2 + \sqrt{(1-\beta^2)(1-4\beta^2)}\right] \]
Second way of doing it
Find velocity of fast ship in rest frame of slow ship
Call this velocity $v'$ and work with $\beta' = \frac{v'}{c}$

\[
\beta' = \frac{2\beta - \beta}{1 - 2\beta^2} = \frac{\beta}{1 - 2\beta^2}
\]

The length of the fast ship in rest frame of slow ship will look contracted as

\[
L''' = \sqrt{1 - \beta'^2} L = \sqrt{1 - \frac{\beta^2}{(1-2\beta^2)^2}} L = \frac{\sqrt{1+4\beta^2 - 5\beta^2}}{1 - 2\beta^2}
\]

Then in this frame the fast ship will need to travel a distance $d = L + L'''$ in order to overtake the other ship. The fast ship is traveling at velocity $v' = \beta'c$

$\Rightarrow$ The time taken will be $t' = \frac{d}{\beta'c}$

\[
t' = \frac{L + L'''}{\beta'c} = \frac{L}{c} \frac{\sqrt{1 - 2\beta^2}}{\beta'c} \left[ 1 + \frac{\sqrt{1+4\beta^2 - 5\beta^2}}{1 - 2\beta^2} \right]
\]

\[
t' = \frac{L}{\beta c} \left[ 1 - 2\beta^2 + \sqrt{1+4\beta^2 - 5\beta^2} \right]
\]

But $\beta c = v$

\[
t' = \frac{L}{v} \left[ 1 - 2\beta^2 + \sqrt{1-5\beta^2 + 4\beta^4} \right]
\]

And this is the method used by many people.
\[ U_x = U \cos 30 = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} c = \frac{\sqrt{3}}{2} \]

\[ U_y = U \sin 30 = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} c = \frac{c}{2\sqrt{3}} \]

Boost into rest frame of B

\[ Y = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{1}{4}}} = \frac{2}{\sqrt{3}} \]

\[ U_x' = \frac{U_x - V}{\sqrt{1 - \frac{u^2}{c^2}}} = 0 \quad \text{since} \quad U_x = V = \frac{\sqrt{3}}{2} \]

\[ U_y' = \frac{U_y}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{\sqrt{3}}{2} \cdot \frac{1}{2\sqrt{3}} = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} = \frac{4}{3} \text{ or } \frac{\sqrt{3}}{3} \]

Then

\[ u' = \sqrt{U_x'^2 + U_y'^2} \Rightarrow \sqrt{\frac{1}{3} c} \]
(4) Use $c=1$, set the appropriate factors of $c$ at the end

Cons. of energy \[ M = \sqrt{m^2 + p^2} + \sqrt{m^2 + p^2} \]

\[ M = 2 \sqrt{m^2 + p^2} \]
\[ M^2 = 4m^2 + 4p^2 \]
\[ p^2 = \frac{M^2 - 4m^2}{4} \]
\[ p = \frac{1}{2} \sqrt{M^2 - 4m^2} \]

Restore factors of $c$ so that \([p] = [m][v]\)

\[ p = \frac{c}{2} \sqrt{M^2 - 4m^2} \]