Poisson 1

Poisson Distribution

event occur at random times

Want $P(N/\mu)$ = probability that N events occur in time interval t where on average μ event occur

λ = prob unit time that event happens

$\Delta p = \lambda \Delta t$ = prob. event occurs in Δt

Let $t = M \Delta t$ where t is some finite time
Δt is very small
M is very large

Probability that no event occurs in time t is

$$P_0(t) = (1 - \lambda \Delta t)^M = \left(1 - \frac{\lambda t}{M}\right)^M$$

$$P_0(t) = 1 + M\left(\frac{-t}{M}\right) + M(M-1)\left(-\frac{t}{M}\right)^2 + M(M-1)(M-2)\left(-\frac{t}{M}\right)^3 + \cdots$$

$$\lim_{M\to\infty} M(M-1) = M^2 \quad \lim_{M\to\infty} M(M-1)(M-2) = M^3 \quad \text{etc}$$

$$\Rightarrow \lim_{M\to\infty} P_0(t) = 1 - t + t^2 - t^4 + \cdots = e^{-\lambda t}$$

$$\boxed{P_0(t) = e^{-\lambda t}}$$

Poisson 2

$$P_1(t+\Delta t) = \underbrace{P_0(t)\cdot\lambda\Delta t}_{\text{event happens btw } t \,\&\, t+\Delta t} + \underbrace{P_1(t)(1-\lambda\Delta t)}_{\text{event happens btw } 0 \text{ and } t}$$

$$P_1(t+\Delta t) = e^{-\lambda t}\lambda\Delta t - \lambda\Delta t\, P_1(t)$$

$$\frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = \lambda e^{-\lambda t} - \lambda P_1(t)$$

$$\frac{dP_1(t)}{dt} = \lambda e^{-\lambda t} - \lambda P_1(t)$$

Solution is $\boxed{P_1(t) = \lambda t\, e^{-\lambda t}}$

Now we show by induction that $P_N(t) = \frac{e^{-\lambda t}(\lambda t)^N}{N!}$

This works for $N=1$ and $N=0$

Assume that it works for N, show that it works for N+1 –

Exactly as before we can show that

$$\boxed{\frac{dP_{N+1}}{dt} = \lambda P_N - \lambda P_{N+1}} \quad (1)$$

Poisson 3

We have assumed $P_N = \dfrac{e^{-\lambda t} (\lambda t)^N}{N!}$

See if $P_{N+1} = \dfrac{e^{-\lambda t} (\lambda t)^{N+1}}{(N+1)!}$ works for eqtn 1

~~Left~~ LHS of eqtn (1)

$$\frac{dP_{N+1}}{dt} = -\frac{\lambda e^{-\lambda t} (\lambda t)^{N+1}}{(N+1)!} + \frac{e^{-\lambda t} \lambda^{N+1} t^N}{N!}$$

$$\frac{dP_{N+1}}{dt} = -\lambda P_{N+1} + \lambda P_N = \text{RHS of eqtn (1)}$$ ✓ OK

Therefor since in time t the mean number of events is $\lambda t = \mu$ we can write

$$\boxed{P(N|\mu) = \frac{e^{-\mu} \mu^N}{N!}}$$

Two properties of Poisson

① If a is poisson random variable
and b is poisson random variable
and (a, b) are uncorrelated
$c = a + b$ is also Poisson
Means $\langle a \rangle = a_0$, $\langle b \rangle = b_0$ $\Rightarrow \langle c \rangle = c_0$

② For large μ, N $\Rightarrow$ N "almost" continous variable (obviously)
$\Rightarrow$ N follows Gaussian PD of mean μ and $\sigma = \sqrt{\mu}$

Proof of ①

$$P(c=k) = \sum_{i=0}^{k} P(b=k-i)\, P(a=i)$$

$$p(c=k) = \sum_{i=0}^{k} e^{-b_0} \frac{b_0^{k-i}}{(k-i)!} e^{-a_0} \frac{a_0^{i}}{i!}$$

$$p(c=k) = e^{-(a_0+b_0)} \sum_{i=0}^{k} \frac{1}{(k-i)!\, i!} b_0^{k-i} a_0^{i}$$

↑ Multiply top & bottom by k!

$$p(c=k) = \frac{e^{-a_0+b_0}}{k!} \sum_{i=0}^{k} \frac{k!}{(k-i)!\,i!} b_0^{k-i} a_0^{i}$$

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$$\boxed{p(c=k) = \frac{e^{-c_0}}{k!} c_0^{k}}$$ where $c_0 = a_0 + b_0$

Proof of 2

$$P(N) = \frac{\mu^N e^{-\mu}}{N!}$$

For large N, Stirling Formula $N! \simeq \sqrt{2\pi N}\, e^{-N} N^N$

$$P(N) \sim \frac{1}{\sqrt{2\pi N}} \left(\frac{\mu}{N}\right)^N e^{N-1}$$

Write $N = \mu(1+\delta)$ $\quad \mu \gg 1 \quad \delta \ll 1$

Then $\frac{\mu}{N} = (1+\delta)^{-1}$ $\quad N - \mu = \mu\delta$ $\quad \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\mu}} (1+\delta)^{-1/2}$

$$P(N) \sim \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\sqrt{\mu}} (1+\delta)^{-1/2}\right] (1+\delta)^{-\mu(1+\delta)} e^{\mu\delta}$$

$$p(N) \sim \frac{e^{\mu\delta}}{\sqrt{2\pi\mu}} \underbrace{(1+\delta)^{-\mu(1+\delta)-1/2}}_{\text{call this term } 1/\alpha} \qquad (2.1)$$

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$$\log\alpha = \left[\mu(1+\delta)+\frac{1}{2}\right]\log(1+\delta)$$

Since δ is small, $\log 1+\delta \simeq \delta - \frac{1}{2}\delta^2$

$$\log\alpha \simeq \left[\mu(1+\delta)+\frac{1}{2}\right]\left[\delta-\frac{1}{2}\delta^2\right] = \mu\delta+\frac{1}{2}\mu\delta^2$$

$$\Rightarrow \frac{1}{\alpha} \cong e^{-\mu\delta}\, e^{-\mu\delta^2/2}$$

Plug this into equation 2.1

$$P(N) \sim \frac{e^{\mu\delta}}{\sqrt{2\pi\mu}}\, e^{-\mu\delta}\, e^{-\mu\delta^2/2} = \frac{e^{-\mu\delta^2/2}}{\sqrt{2\pi\mu}}$$

Going back to $N=\mu(1+\delta)$ $\quad \delta = \frac{N-\mu}{\mu}$

$$\boxed{P(N)\sim \frac{1}{\sqrt{2\pi\mu}}\, e^{-(N-\mu)^2/2\mu}}$$

Gaussian of mean μ and $\sigma = \sqrt{\mu}$