DISCRETE FOURIER TRANSFORM (DFT)

Most often used to go from time domain to frequency domain -

For periodic function, period T -

Fourier Series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

$$\omega_n = n\frac{2\pi}{T} \leftarrow \text{discrete}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\omega_n t\, dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\omega_n t\, dt$$

Letting $t \to \infty$, ω becomes a continous variable

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

FOURIER TRANSFORM FT

Also used $e^{i\omega t} = \cos\omega t + i\sin\omega t$

$F(\omega)$ gives the amplitude of the sin-wave of frequency ω in the decomposition of $f(t)$ - Note $F(\omega)$ is complex (will come back to that later)

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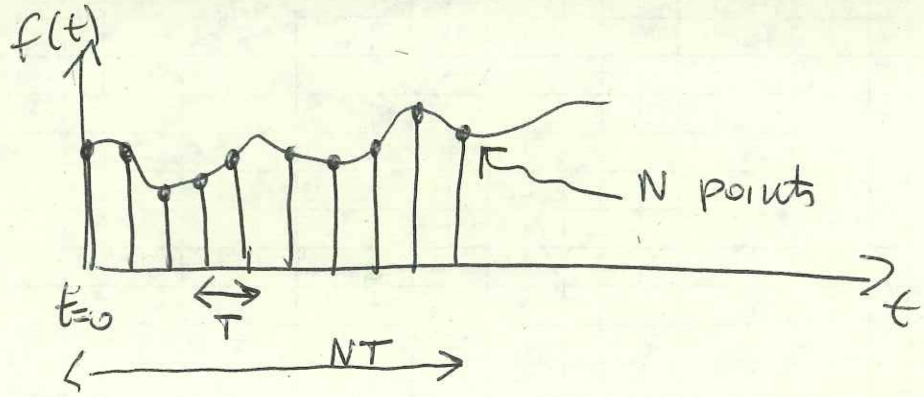
DFT is approx to FT when $f(t)$ is

- sampled over finite range
- sampled at discrete times



We have $f(t=0), f(t=T), f(t=2T), \dots f(t=(N-1)t)$

Short hand $f(0), f(1), \dots f(N-1)$

$$F(\omega) \rightarrow \sum_{k=0}^{N-1} f(k)\, e^{-i\omega k T}$$

Since $f(t)$ is only sampled N times, makes sense to also sample ω ~~for if~~

Frequencies to be sampled

$$f = 0, \frac{1}{NT}, \frac{2}{NT}, \dots, \frac{(N-1)}{NT}$$

↑

DC

$$f_n = \frac{n}{NT} \qquad n = 0 \rightarrow N-1$$

$$\omega_n = 2\pi f_n = \frac{2\pi n}{NT}$$

$$F(\omega) \rightarrow F(n) = \sum_{k=0}^{N-1} f(k)\, e^{-i\frac{2\pi n k}{N}}$$

The inverse is $\quad f(k) = \frac{1}{N} \sum_{n=0}^{N-1} F(n)\, e^{+i\frac{2\pi n k}{N}}$

So $\frac{1}{N} F(N)$ is the (complex) amplitude for frequency ω_n –

~~Fio~~ Imagine $f(t) = \text{constant} = C$ (DC)

$$F(0) = \sum_{k=0}^{N} C = CN$$

DC amplitude is $\frac{F(0)}{N} = C \quad \checkmark$ OKAY

~~Now, notes c — N~~

(Drop the $f(t) = C$ assumption)

More generally

$$F(1) = \sum_{k=0}^{N-1} f(k)\, e^{-i\frac{2\pi k}{N}} = \sum_{k=0}^{N-1} f(k) \left[\cos\frac{2\pi k}{N} - i \sin\frac{2\pi k}{N}\right]$$

Consider $F(N-1)$

The exponent will be

$$-i\frac{2\pi k(N-1)}{N} = i 2\pi k + i\frac{2\pi k}{N}$$

But $e^{2\pi i k} = 1$ if k is integer

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$$F(N-1) = \sum_{k=0}^{N-1} f(k)\, e^{+i\frac{2\pi k}{N}}$$

Same as F(1) but for sign of exponent

$$F(N-1) = \sum_{k=0}^{N-1} f(k) \left[\cos\frac{2\pi k}{N} + i \sin\frac{2\pi k}{N} \right]$$

Then $\quad F(1) + F(N-1) = \sum_{k=0}^{N-1} 2\, f(k) \cos\frac{2\pi k}{N}$

F(1) and F(N-1) have same frequency info

Same for

F(2) - F(N-2)

F(3) - F(N-3) etc.

The info on the ~~...~~ frequency resides only in the first ½ of the F's.

The amplitude corresponding to the n-th frequency is

$$A_n = |F(n)|$$

remember

F(n) is complex!