Here are some notes on the "direct method" and the "acceptance/rejection" method for generating random numbers x according to a pdf $f(x)dx$ starting from a uniform random number generator between 0 and 1.

For the direct method the recipe is:

- Calculate the cumulative distribution
  $F(x) = \int_{-\infty}^{x} f(x')dx'$
- Set $R = F(x)$
- Solve for $x(R)$
- Generate a bunch of uniform random numbers R
- For each R calculate $x(R)$
- You now have a bunch of x's which follow a pdf $f(x)dx$

Full disclosure: these are not by notes (except for the 1st page). They were given to me when I was a graduate student, and I have somehow kept them all this time.

WANT TO GENERATE A VARIABLE x DISTRIBUTED ACCORDING TO A PROBABILITY DISTRIBUTION $f(x)\,dx$ -

Let us pick RANDOM NUMBER R DISTRIBUTED ~~[illegible]~~ UNIFORMLY BETWEEN 0 AND 1, (i.e. ITS PROBABILITY DISTRIBUTION $P(R)dR$ is SUCH THAT $P(R)=1$).

WHAT SHOULD BE $x(R)$ SUCH THAT x is DISTRIBUTED AS $f(x)\,dx$?

START WITH $P(R)\,dR$

CHANGE VARIABLES TO $X(r)$ $\Rightarrow$ $P(R)\,dR = dR = \frac{dR}{dX}dX$

SO $\frac{dR}{dX}dx$ is THE PROBABILITY DISTRIBUTION FOR x

WE WANT $\frac{dR}{dX} = f(x)$ $\Longrightarrow$ $R = \int_0^x f(x')\,dx'$

LETS DEFINE $F(x) = \int_0^x f(x')\,dx'$

THEN $R = F(x)$ $\Longrightarrow$ $\boxed{x = F^{-1}(R)}$

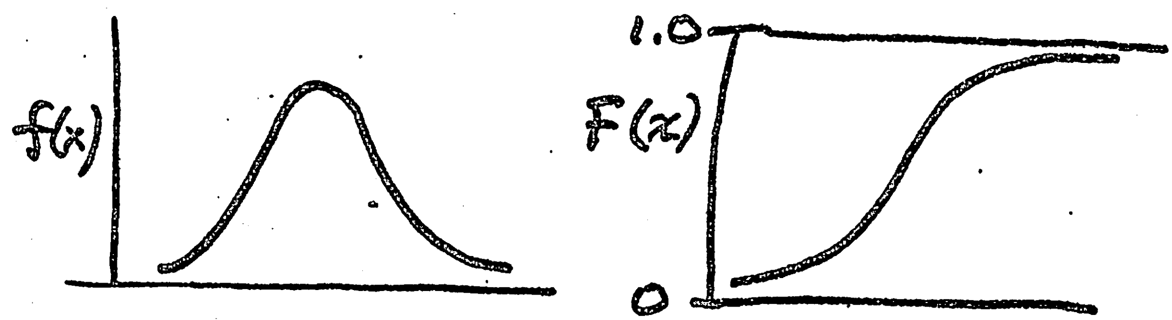
Monte Carlo Methods

A. Direct Method for Continuous Random Variable

We want to generate x for which the distribution function is $f(x)$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

We define the cumulative distribution function

$$F(x) = \int_0^x f(x')\,dx'$$



The direct method says

$$F(x) = R$$

$$x = F^{-1}(R)$$

where R is a random number

$$f(R) = \begin{cases} 1, & 0 \le R \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

An Example ...

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad 0 \le x \le \infty$$

$$F(x) = 1 - e^{-x/\lambda}$$

set $1 - e^{-x/\lambda} = R' = 1 - R$

$\boxed{\text{Then } x = -\lambda \ln R}$

More General

$$f(x) = A e^{-x/\lambda} \qquad x_1 \le x \le x_2$$

$$x = -\lambda \ln\left[-\left(e^{-x_2/\lambda} - e^{-x_1/\lambda}\right)R + e^{-x_2/\lambda}\right]$$

More Examples of Direct Method

$f(x)$	Range	Solution
$\frac{n-1}{x^n}$, $n>1$	$0,\infty$	$x = R^{\frac{1}{1-n}}$
$\frac{1}{x^2}$	$0,\infty$	$x = 1/R$
$(n+1)x^n$ $n \neq -1$	$0,1$	$x = R^{\frac{1}{n+1}}$
$3x^2$	$0,1$	$x = \sqrt{R}$
$\frac{\sin\theta}{2}$	$0,\pi$	$\cos\theta = 1-2R$
$\frac{1}{\pi}\frac{\Gamma/2}{(E-E_0)^2+(\Gamma/2)^2}$	$-\infty,\infty$	$E = E_0 + \frac{\Gamma}{2}\tan\left[\pi\left(R-\frac{1}{2}\right)\right]$

Limitations of the direct method

1. Must know functional form of $f(x)$ or $F(x)$
2. Must be able to integrate $f(x)$ if $F(x)$ not known by itself.
3. Must be able to invert $F(x)$.

B. The Acceptance - Rejection method does not need items 2 and 3 above.

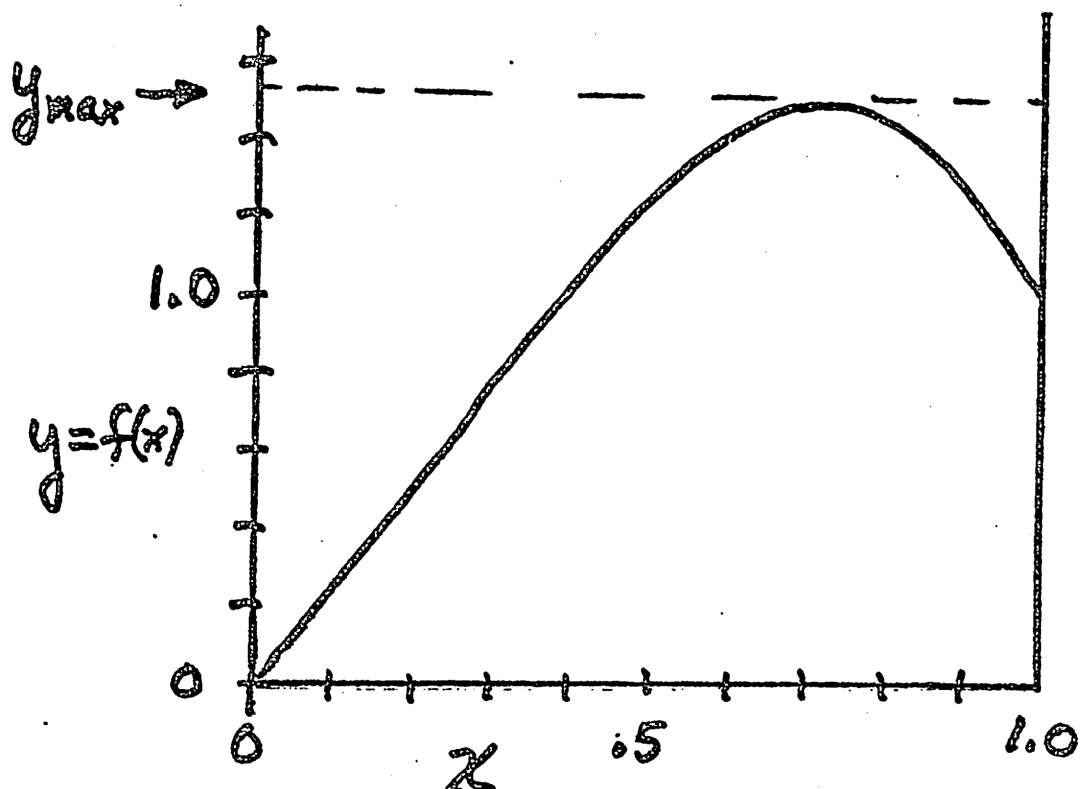
Consider a simple example

$$f(x) = 2x + 3x^2 - 4x^3$$

$$F(x) = x^2 + x^3 - x^4$$

The equation $F(x) = R$ is time consuming to solve

$f(x) = 2x + 3x^2 - 4x^3$



Aceeptance-Rejection Method

1. Choose x : $x = R_1$
2. Choose y : $y = y_{max} R_2$
3. Keep x if (x, y) is under $f(x)$

Same thing in different words

1. Choose x : $x = R_1$
2. Assign a weight $W = f(x)/y_{max}$
3. Reject if $R_2 > W$

C. Combination of direct and Acceptance-Rejection Methods

Example: $f(x) = A\, x^{1/2} e^{-3x/2}$

$0 \leq x < \infty$

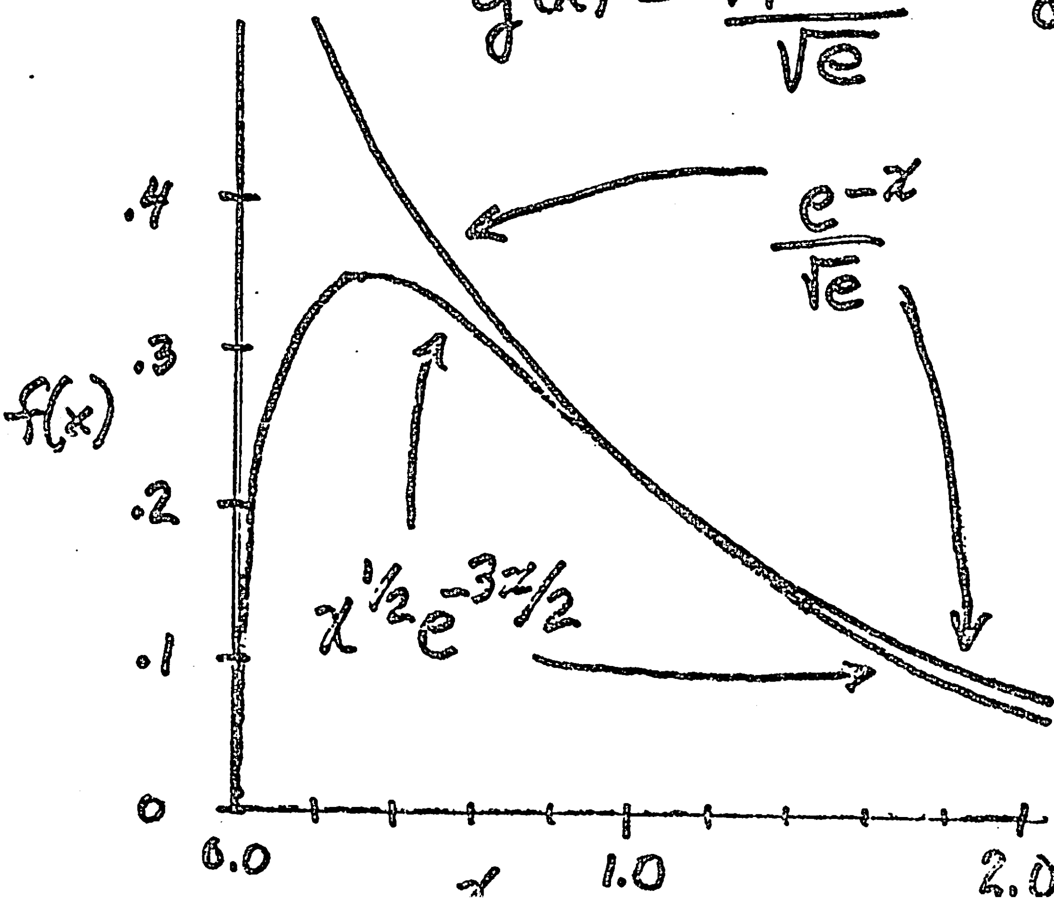
The direct method would be difficult at best (I don't know how.)

Simple Acceptance-Rejection is impossible

Look for a function with a finite area that is always larger than $f(x)$

In this case

$g(x) = \dfrac{A e^{-x}}{\sqrt{e}}$ does this



Procedure:

1. Generate x according to
$f(x) = e^{-x}$, $x = -\ln R_1$

2. Keep this if $f(x) > \dfrac{R_2 e^{-x}}{\sqrt{e}}$.

The efficiency of this method (fraction of attempts accepted)

is $\sqrt{e}A = \dfrac{\sqrt{e\pi}}{2\,(3/2)^{3/2}} = 0.795$

Special Cases

1. Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Can be generated in pairs

$$\int f(x) f(y) = \frac{1}{2\pi}\iint e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \int_0^\infty r e^{-r^2/2}\,dr \int \frac{d\phi}{2\pi}$$

for $\mu = 0,\ \sigma = 1$

This can be integrated and inverted to give

$$r = \sqrt{-2\ln R_1}$$

$$\phi = 2\pi R_2$$

and

$$R_{G1} = \mu + \sigma r \sin\phi$$

$$R_{G2} = \mu + \sigma r \cos\phi$$

R_{G1} and R_{G2} are independent gaussian random numbers.