Another way of understanding MC integration

Consider some pdf $p(x)\,dx$ over an interval $[a, b]$.

Imagine that we pick some values of $x$ according to this pdf: $x_1, \ldots, x_N$, \{x\}_i

Now imagine that there is some other function $h(x)$. I take these \{x\}_i and plug them into the function $h(x)$

I then get a set \{h\}_i with $h_i = h(x_i)$

Consider $S = \sum_{i=1}^{N} h_i$ and $T = \frac{1}{N} S$

$T = \frac{1}{N} \sum_{i=1}^{N} h_i = \langle h_i \rangle \quad \text{average} \; h_i$

Now imagine that I take the $x_i$ and sort them into bins of width $\Delta x$.

Kind of like putting them into a histogram.

\[\text{This function will be proportional to } p(x)\,dx\]

Each bin $j$ contains $N p(x) \Delta x$ entries

The number of bins is $N$
Each bin \( j \) will then contribute an amount

\[ NP(x_j) \Delta x \cdot h(x_j) \]

to the sum \( S \)

\[ \Rightarrow S = \sum_{j=1}^{M} NP(x_j) h(x_j) \Delta x \]

Letting \( \Delta x \to dx \) this becomes

\[ S = N \int_{a}^{b} p(x) h(x) \, dx \]

But we also had \( S = NT = N \langle h_i \rangle \)

\[ \Rightarrow \int_{a}^{b} p(x) h(x) \, dx = \langle h_i \rangle \]  \[ \square \]

where the \( h_i \) are obtained starting from a set of \( x_i \) picked according to \( p(x) \, dx \) and such that \( h_i = h(x_i) \)

\[ \frac{1}{N} \]

Now look at

\[ \int_{a}^{b} f(x) \, dx = (b-a) \int_{a}^{b} f(x) \, \frac{dx}{b-a} \]

\[ \int f(x) \, dx = (b-a) \int_{a}^{b} f(x) p(x) \, dx \]

where \( p(x) = \frac{1}{b-a} \, dx \) is a pdf which is uniform between \( a \) and \( b \).
Therefore the simple MC method is for $I = \int_a^b f(x) \, dx$ is the following:

1. Pick $\{x_i\}$ uniform between $b$ and $a$
2. Calculate $\{f_i\}$ as $f_i = f(x_i)$
3. Find the average $\langle f_i \rangle$
4. $I \approx (b-a) \langle f_i \rangle = \frac{b-a}{N} \sum_{i=1}^{N} f_i$

This can be generalized to more dimension $N$.

How accurate is this? Depends on how precise the measurement of $\langle f_i \rangle$ is.

Variance of $f_i$: $\sigma_f^2 = \frac{1}{N-1} \sum_{i=1}^{N} [f_i - \langle f_i \rangle]^2$

Then since $I = \frac{b-a}{N} \sum_{i=1}^{N} f_i$, propagation of errors gives

$$\sigma_I^2 = \left(\frac{b-a}{N}\right)^2 \sigma_f^2 = \frac{(b-a)^2}{N^2} \left( N \sigma_f^2 \right)$$

$$\Rightarrow \sigma_I = \frac{b-a}{\sqrt{N}} \sigma_f$$
The more complicated MC method is for

\[ I = \int_a^b h(x) \, p(x) \, dx \]

where \( p(x) \, dx \) is normalized PDF

1. pick \( \{x_i\} \) according to \( p(x) \, dx \)
2. Calculate \( h_i = h(x_i) \)
3. \( I \approx \langle h_i \rangle \)

Importance sampling

\[ I = \int_a^b f(x) \, dx \] Rewrite it as

\[ I = \int_a^b h(x) \, p(x) \, dx \]

with \( h(x) = \frac{f(x)}{p(x)} \) - Then use this method

Why is this better?

It is better if you can choose \( p(x) \) that looks like \( f(x) \), so that \( h(x) = \frac{f(x)}{p(x)} \) does not change very much as a function of \( x \). This then gives more precise \( \langle h \rangle \)