Another way of understanding MC integration

Consider some pdf $p(x)dx$ over an interval $[a,b]$.

Imagine that we pick some values of x according to this pdf : $x_1 \ldots x_N$, $\{x_i\}$

N of these

Now imagine that there is some other function $h(x)$. I take these $\{x_i\}$ and plug them into the function $h(x)$

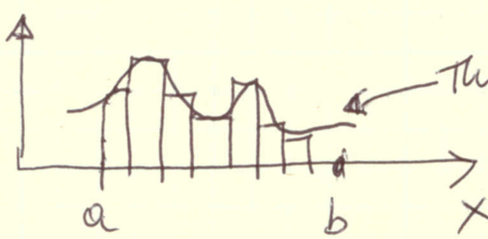
I then get a set $\{h_i\}$ with $h_i = h(x_i)$

Consider $$S = \sum_{i=1}^{N} h_i \quad \text{and} \quad T = \frac{1}{N} S$$

$$T = \frac{1}{N} \sum_{i=1}^{N} h_i = \langle h_i \rangle$$ average h_i

Now ~~that~~ imagine that I take the x_i and sort them into bins of widt Δx.

Kind of Like putting them into a histogram.



This function will be proportional to $p(x)dx$

Each bin j contains $N p(x) \Delta x$ entries

The number of bins is M

Each bin j will then contribute an amount

$N p(x_j) \Delta x \cdot h(x_j)$ to the sum S

$$\Rightarrow S = \sum_{j=1}^{M} N p(x_j) h(x_j) \Delta x$$

Letting $\Delta x \to dx$ this becomes

$$S = N \int_a^b p(x) h(x) \, dx$$

But we also had $S = NT = N \langle h_i \rangle$

$$\Rightarrow \boxed{\int_a^b p(x) h(x) \, dx = \langle h_i \rangle} \quad (1)$$

where the h_i are obtained starting from a set of x_i picked according to $p(x)\,dx$ and such that $h_i = h(x_i)$

Now look at $\int_a^b f(x) \, dx = (b-a) \int_a^b f(x) \frac{dx}{b-a}$

$$\int f(x) \, dx = (b-a) \int_a^b f(x) p(x) \, dx$$

Where $p(x) = \frac{1}{b-a} dx$ is a pdf which is uniform between a and b.

Therefore the simple MC method for $I = \int_a^b f(x)\,dx$ is the following

(1) Pick $\{x_i\}$ uniform between b and a

(2) Calculate $\{f_i\}$ as $f_i = f(x_i)$

(3) Find the average $\langle f_i \rangle$

(4) $$I \cong (b-a)\langle f_i \rangle = \frac{b-a}{N} \sum_{i=1}^{N} f_i$$

This can be generalized to more dimension

How accurate is this? Depends on how precise the measurement of $\langle f_i \rangle$ is

Variance of f_i $\quad \sigma_f^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left[f_i - \langle f_i \rangle\right]^2$

Then since $I = \frac{b-a}{N} \sum_{i=1}^{N} f_i$ propagation of errors gives

$$\sigma_I^2 = \left(\frac{b-a}{N}\right)^2 \sum_{i=1}^{N} \sigma_f^2 = \frac{(b-a)^2}{N^2}\left(N \sigma_f^2\right)$$

$$\Rightarrow \boxed{\sigma_I = \frac{b-a}{\sqrt{N}} \sigma_f}$$

The more complicated MC method is for

$$I = \int_a^b h(x)\, p(x)\, dx$$

where $p(x)dx$ is normalized pdf

(1) pick $\{x_i\}$ according to $p(x)dx$

(2) Calculate $h_i = h(x_i)$

(3) $I \cong \langle h_i \rangle$

Importance sampling

$I = \int_a^b f(x)\, dx$ – Rewrite it as

$\cancel{I = \int_a^b h(x)\, dx}$ $\int_a^b h(x)\, p(x)\, dx$

with $h(x) = \dfrac{f(x)}{p(x)}$ – Then use this method

Why is this better?

It is better if you can ~~pick~~ choose a $p(x)$ that looks like $f(x)$, so that $h(x) = \dfrac{f(x)}{p(x)}$ does not change very much as a function of x - This then gives more precise $\langle h \rangle$