Cross Products: \((\vec{a} \times \vec{b})_k = \epsilon_{ijk}a_ib_j\).

Helpful Integrals: \( \int d^3x = \int_0^\infty r^2dr \int d\Omega, \int d\Omega = \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi = 4\pi, \int_0^\infty r^ne^{-r/a} \, dr = n!a^{1+n} \).

Schrödinger Equation: \(ih\frac{\partial}{\partial t}\vert \Psi \rangle = \hat{H}\vert \Psi \rangle w/\) Hamiltonian \(\hat{H} = \frac{\vec{p}^2}{2m} + V\).

Harmonic oscillator (1d): For \(\hat{H} = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2x^2\), raising/lowering operators \(a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p})\), \(\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\), \(\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(a_+ - a_-)\), \([a_-, a_+] = 1\), \(\hat{H} = \hbar\omega(a_+a_+ + 1/2)\), \(E_n = \hbar\omega(n + 1/2)\), \(a_+\psi_n = \sqrt{n+1}\psi_{n+1}\), \(a_-\psi_n = \sqrt{n}\psi_{n-1}\), \(\psi_0(x) = (\frac{\hbar}{4\pi})^{1/4} e^{-\frac{\hbar}{2}\pi x^2}\).

Laplacian: \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) (Cartesian), \(\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial \phi^2}\right)\) (spherical).

QM in 3D: Position operator \(\hat{x} = (x, y, z)\) and momentum operator \(\hat{p} = (p_x, p_y, p_z)\) in Cartesian coords. Position space \(p_x = -ih\frac{\partial}{\partial x}, p_y = -ih\frac{\partial}{\partial y}, p_z = -ih\frac{\partial}{\partial z}\), so \(\hat{p} = -i\hbar\nabla\) and \(\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V\). Commutators \([x, p_x] = [y, p_y] = [z, p_z] = i\hbar\), all other commutators of \(x, y, z, p_x, p_y, p_z\) are zero.

Spherically symmetric potentials: \(V(r) = V(r)\), eigenstates \(\psi_{n, \ell, m} = R_{n, \ell}(r)Y^{m}_{\ell}(\theta, \phi)\), radial momentum \(\hat{r} = -ih(\frac{\partial}{\partial r} + \frac{1}{r})\) and \(\hat{p}^2 = -\hbar^2\frac{1}{r^2}(\frac{\partial^2}{\partial r^2})\).

Radial equation: \(\hat{H}_\ell R_\ell(r) = \left[\frac{\vec{p}^2}{2m} + V(r) + \frac{\hbar^2(\ell+1)}{2mr^2}\right] R_\ell(r) = E_\ell R_\ell(r)\).

Harmonic oscillator (3d): \(\hat{H} = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2r^2\). Operator \(\hat{a}_\ell = \frac{\sqrt{2m\hbar\omega}}{\sqrt{\hbar\omega}^2}(i\hat{p}_r + m\omega r - \frac{(\ell+1)\hbar}{r}); \hat{H}_\ell = \hbar\omega(\hat{a}^\dagger_\ell\hat{a}_\ell + l + 3/2)\). \(\hat{a}_\ell\) raises \(\ell\) by one and lowers \(E_\ell\) by \(\hbar\omega\) while \(\hat{a}^\dagger_\ell\) does the opposite. \([\hat{a}_\ell, \hat{a}^\dagger_\ell] = \frac{\hbar^2}{\hbar\omega}(\hat{H}_{\ell+1} - \hat{H}_\ell) + 1\).

Spherical Harmonics: \(Y^m_{\ell}(\theta, \phi)\) orthogonal in both \(\ell, m\). Simplest harmonics:

\[
Y^0_0 = \left(\frac{1}{4\pi}\right)^{1/2}, \quad Y^0_1 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \quad Y^\pm_1 = \mp \left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}.
\]

Hydrogen atom: \(V(r) = -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{r}\), energies \(E_n = -\frac{\hbar^2}{2a_0^2n^2} = -\mathcal{R}/n^2\), Rydberg constant \(\mathcal{R} = 13.6\) eV, Bohr radius \(a_0 \equiv \frac{4\pi\varepsilon_0\hbar^2}{m\varepsilon_0} = 0.53\) Å, ground state wavefunction \(\psi_{1, 0, 0}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_0}} e^{-r/a_0}\) Operator \(\hat{a}_\ell = \frac{a_0}{\sqrt{2\hbar}}(i\hat{p}_r + \frac{\hbar}{a_0(\ell+1)} - \frac{(\ell+1)\hbar}{r}); \hat{H}_\ell = \frac{\hbar^2}{ma_0^2}(\hat{a}^\dagger_\ell\hat{a}_\ell - \frac{1}{(\ell+1)^2}); \hat{a}_\ell\) raises \(\ell\) by one while \(\hat{a}^\dagger_\ell\) does the opposite. \([\hat{a}_\ell, \hat{a}^\dagger_\ell] = \frac{ma_0^2}{\hbar^2}(\hat{H}_{\ell+1} - \hat{H}_\ell)\).

Angular Momentum: \([\hat{L}_x, \hat{L}_y] = ih\varepsilon_{ijk}\hat{L}_k\); \(\hat{L}_x^2, \hat{L}_y, \hat{L}_z, \hat{L}_x^2, \hat{L}_y, \hat{L}_z\) are the basis vectors.

Raising and lowering operators: \(\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y\) for eigenstates of \(\hat{L}^2\) and \(\hat{L}_z, \hat{L}^\pm\). Then: \(\hat{L}_\pm |\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)} |\ell, m\rangle\) and \(\hat{L}_\mp |\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)} |\ell, m-1\rangle\). In terms of these operators, \(\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)\) and \(\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-).

Matrix operators: \(M_{ij} = \langle u_i | \hat{M} | u_j \rangle\) where the \(|u_k\rangle\)'s are the basis vectors.

Spin operators:

\[
\hat{S}_x = \frac{\hbar}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \hat{S}_y = \frac{\hbar}{2} \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \hat{S}_z = \frac{\hbar}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\]