Cross Products: $(\vec{a}\times\vec{b})_k = \epsilon_{ijk}a_i b_j$.

Helpful Integrals: $\int d^3x = \int_0^\infty r^2 dr \int d\Omega$, $\int d\Omega = \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$, $\int_0^\infty r^n e^{-r/a}\, dr = n!\, a^{1+n}$.

Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$ w/ Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V$

Harmonic oscillator (1d): For $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, raising/lowering operators $a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p})$, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$, $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$, $[a_-, a_+] = 1$, $\hat{H} = \hbar\omega(a_+a_- + 1/2)$, $E_n = \hbar\omega(n+1/2)$, $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\psi_{n-1}$, $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$.

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (Cartesian), $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$ (spherical).

QM in 3D: Position operator $\vec{x} = (x, y, z)$ and momentum operator $\vec{p} = (p_x, p_y, p_z)$ in Cartesian coords. Position space $p_x = -i\hbar\frac{\partial}{\partial x}, p_y = -i\hbar\frac{\partial}{\partial y}, p_z = -i\hbar\frac{\partial}{\partial z}$, so $\vec{p} = -i\hbar\nabla$ and $H = -\frac{\hbar^2}{2m}\nabla^2 + V$. Commutators $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$, all other commutators of x, y, z, p_x, p_y, p_z are zero.

Spherically symmetric potentials: $V(\vec{r}) = V(r)$, eigenstates $\psi_{n,\ell,m} = R_{n,\ell}(r)Y_\ell^m(\theta,\phi)$, radial momentum $\hat{p}_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$ and $\hat{p}_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$.

Radial equation: $\hat{H}_\ell R_\ell(r) = \left[\frac{\hat{p}_r^2}{2m} + V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]R_\ell(r) = E_\ell R_\ell(r)$.

Harmonic oscillator (3d): $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{r}^2$. Operator $\hat{a}_\ell = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p}_r + m\omega r - \frac{(\ell+1)\hbar}{r})$; $\hat{H}_\ell = \hbar\omega(\hat{a}_\ell^\dagger\hat{a}_\ell + l + 3/2)$. $\hat{a}_\ell$ raises ℓ by one and lowers E_ℓ by $\hbar\omega$ while $\hat{a}_\ell^\dagger$ does the opposite. $[\hat{a}_\ell, \hat{a}_\ell^\dagger] = \frac{\hat{H}_{\ell+1} - \hat{H}_\ell}{\hbar\omega} + 1$.

Spherical Harmonics: $Y_\ell^m(\theta,\phi)$ orthogonal in both ℓ, m. Simplest harmonics:

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \qquad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$$

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Hydrogen atom: $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$, energies $E_n = -\frac{\hbar^2}{2ma_0^2n^2} = -\mathcal{R}/n^2$, Rydberg constant $\mathcal{R} = 13.6$ eV, Bohr radius $a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.53$ Å, ground state wavefunction $\psi_{1,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$, Operator $\hat{a}_\ell = \frac{a_0}{\sqrt{2}\hbar}(i\hat{p}_r + \frac{\hbar}{a_0(\ell+1)} - \frac{(\ell+1)\hbar}{r})$; $\hat{H}_\ell = \frac{\hbar^2}{ma_0^2}(\hat{a}_\ell^\dagger\hat{a}_\ell - \frac{1}{2(\ell+1)^2})$. $\hat{a}_\ell$ raises ℓ by one while $\hat{a}_\ell^\dagger$ does the opposite. $[\hat{a}_\ell, \hat{a}_\ell^\dagger] = \frac{ma_0^2}{\hbar^2}(\hat{H}_{\ell+1} - \hat{H}_\ell)$.

Angular Momentum: $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$. $\hat{L}^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle$, $\hat{L}_z|\ell,m\rangle = \hbar m|\ell,m\rangle$.

Raising and lowering operators: $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ for eigenstates of $\hat{L}^2$ and $\hat{L}_z$. Then: $\hat{L}_+|\ell,m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|\ell$ and $\hat{L}_-|\ell,m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|\ell, m-1\rangle$. In terms of these operators, $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$ and $\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$.

Matrix operators: $M_{ij} = \left\langle u_i \middle| \widehat{M} | u_j \right\rangle$ where the $|u_k\rangle$'s are the basis vectors.

Spin operators:

$$\widehat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \widehat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \widehat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$