Matrix representation of operators

Complete set of orthonormal basis states $|u_i\rangle$

Usually eigenstates of hermitian operator

$\langle u_i | u_j \rangle = \delta_{ij}$

Any state $|\Psi\rangle$ can be written as $|\Psi\rangle = \sum |u_i\rangle \phi_i$

where the sum over repeated indices is implied and $\phi_i$'s are just numbers. Let $O$ be some operator

$O |\Psi\rangle = \sum \phi_i O |u_i\rangle$ (1)

But $O |\Psi\rangle = \sum \phi_i |u_i\rangle$ (2)

Acting with $O$ on $O |\Psi\rangle$ we find

$\phi_i = \langle u_i | O |\Psi\rangle = \sum \phi_j \langle u_i | O |u_j\rangle = O_{ij} \phi_j$

with $O_{ij} = \langle u_i | O |u_j\rangle$

This is a matrix equation

$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} O_{11} & O_{12} & \cdots \\ O_{21} & O_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix}$

Therefore the operator $O$ can be represented by a matrix $O_{ij} = \langle u_i | O |u_j\rangle$