Addition of Angular Momentum

Nathaniel Craig

1 Addition of angular momentum

You have now learned about the quantum mechanical analogue of angular momentum, both the familiar extrinsic angular momentum corresponding to the operator $\mathbf{L}$, and a completely new intrinsic angular momentum quantity, *spin*, corresponding to the operator $\mathbf{S}$. Both are generators of rotations – the former rotations in three spatial dimensions, the latter rotations in the "internal" space of spins – and so it follows that they have the same algebraic structure of commutators.

It is all well and good to have understood the orbital or spin angular momentum properties of a given state, but more generally we will be interested systems where we combine different types of angular momentum, and would like to relate the total angular momentum of a system to the angular momenta of its constituents. Classically we can add angular momenta, and quantum mechanically we can do so as well. There are generally two situations of interest:

- The total angular momentum of a single particle, i.e. its combined spin and orbital angular momentum. We will usually denote this with the operator $\vec{J}$, to distinguish it from $\vec{L}$ and $\vec{S}$, with the definition $\vec{J} = \vec{L} + \vec{S}$. In studying the hydrogen atom, we have learned to label the orbital wavefunctions of the electron by their orbital angular momentum. But electrons have spin, so if we are careful we should be able to describe the total angular momentum of the electron in terms of the sum of its spin and orbital angular momentum.

- The total spin or total orbital angular momentum of two (or more) particles. Consider again the hydrogen atom – in solving the hydrogen atom we ignored the dynamics of the proton to find the electron wavefunction, which is perfectly fine because the proton is so much heavier than the electron and can be treated like a static source. But if we are interested in the total spin of the hydrogen atom, we need to account for the spins of both the electron and the proton.

In each case we would like to relate the total angular momentum to the angular momentum of the two constituents.

2 Addition of spins

Let's start by asking what happens if we have two distinct particles of spin s_1 and s_2, and we want to figure out the total spin of this system. As an example, you can think of the total spin of the hydrogen atom, which is the sum of spins of the proton and the electron. (In this case, consider them in the ground state so we can neglect orbital angular momentum.)

We are interested in the operator that consists of the sum of the spin operators for the two particles,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

Now this is the first time we have attempted to carefully treat a quantum mechanical system involving two particles. We will devote much more time to this shortly, but for the time being, if the two particles are distinct and the properties of the first particle are entirely independent of the properties of the second particle, you should understand this to mean that the Hilbert space factorizes into products of states, $|1\rangle \otimes |2\rangle$, where $\otimes$ is the *Kronecker product*. The spin operator acting on this product factorizes as well, i.e.

$$\mathbf{S} = \mathbf{S}_1 \otimes I + I \otimes \mathbf{S}_2$$

i.e. $\mathbf{S}_1$ acts only on the first particle, and $\mathbf{S}_2$ acts only on the second particle. There are some more general subtleties that we will encounter when the particles are indistinguishable, but we'll discuss that in the next few lectures.

We can go ahead and form raising and lowering operators for the total spin $\mathbf{S}$ much as before,

$$S_\pm = (S_{1x} + iS_{1y}) + (S_{2x} + iS_{2y})$$

as well as the square of the total angular momentum, S^2, the z-component, S_z, etc.

So far, so good. Now we'd like to figure out how the spin states of the two particles are related to the total spin of the system. Imagine, for example, that particle 1 is in the spin state $|s_1\, m_1\rangle$ (where s_1 is its eigenvalue with respect to $(S_1)^2$ and m_1 is its eigenvalue with respect to S_{1z}), and particle 2 is in the spin state $|s_2\, m_2\rangle$ with respect to $\mathbf{S}_2$. We denote this total state as $|s_1\, m_1\rangle|s_2\, m_2\rangle$ (or more explicitly

$|s_1\, m_1\rangle \otimes |s_2\, m_2\rangle$).

What is the total spin of this state, and its spin in the z direction? That is to say,

$$|s\, m\rangle \underset{?}{\longleftrightarrow} |s_1\, m_1\rangle|s_2\, m_2\rangle$$

Perhaps your first thought is that this problem must be trivial – i.e., we can just algebraically solve for s, m in terms of s_1, m_1, s_2, and m_2. This would be true if $S^2, S_z, (S^1)^2, S_{1z}, (S^2)^2$, and S_{2z} formed a complete set of commuting observables. But they do not! We have

$$S^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \quad (1)$$

$$S_z = S_{1z} + S_{2z} \quad (2)$$

(In the first line we have used the fact that $\mathbf{S}_1$ and $\mathbf{S}_2$ commute.) From this we can see that $m = m_1 + m_2$, but s is less clear. In particular, we can see clearly that S^2 does not commute with $\mathbf{S}_1$ or $\mathbf{S}_2$ individually. For instance,

$$[S_{1z}, S^2] = 2[S_{1z}, \mathbf{S}_1 \cdot \mathbf{S_2}] = 2[S_{1z}, S_{1x}S_{2x} + S_{1y}S_{2y}] = 2i\hbar(S_{1y}S_{2x} - S_{1x}S_{2y})$$

is nonzero. This tells us that S_{1z} and S^2 do not share a complete set of eigenstates. So if we want to relate total spin to the spins of each particle, we are going to have to go between two different sets of eigenstates.

What are the two good sets of eigenstates? That is to say, what are good sets of commuting observables for this system? By construction, the spins of the two individual particles do indeed form a complete set of commuting operators, so we know S_1^2, S_{1z}, S_2^2, and S_{2z} are complete, and we can describe the system in terms of s_1, s_2, m_1, m_2. This is sometimes called the "uncoupled representation", since it describes the spins of the two particles separately.

What about a set of observables involving S^2, S_z? The first complete set had four quantities associated with it, so you might expect we can add two operators to S^2, S_z to form a complete set. A sensible choice is S_1^2, S_2^2, since e.g.

$$[S_1^2, S^2] = 2[S_1^2, \mathbf{S}_1 \cdot \mathbf{S_2}] = 0$$

and you can check for yourself that all other commutators vanish. The relevant quantum numbers are s, m, s_1, s_2; this is sometimes called the "coupled representation" since it describes the total spin of the system. This also forms a complete set of

commuting observables. Thus we expect that we can describe this system in terms of

$$S_1^2, S_{1z}, S_2^2, S_{2z} \qquad \text{or} \qquad S^2, S_z, S_1^2, S_2^2$$

with corresponding quantum numbers s_1, s_2, m_1, m_2 or s, m, s_1, s_2. To be super explicit, in terms of operators, eigenstates, and eigenvalues, we have in the former case

$$\begin{bmatrix} S_1^2 \\ S_{1z} \\ S_2^2 \\ S_{2z} \end{bmatrix} |s_1\, m_1\rangle|s_2\, m_2\rangle = \hbar^2 \begin{bmatrix} s_1(s_1+1) \\ m_1/\hbar \\ s_2(s_2+1) \\ m_2/\hbar \end{bmatrix} |s_1\, m_1\rangle|s_2\, m_2\rangle \tag{3}$$

and in the latter case

$$\begin{bmatrix} S^2 \\ S_z \\ S_1^2 \\ S_2^2 \end{bmatrix} |s\, m\, s_1\, s_2\rangle = \hbar^2 \begin{bmatrix} s(s+1) \\ m/\hbar \\ s_1(s_1+1) \\ s_2(s_2+1) \end{bmatrix} |s\, m\, s_1\, s_2\rangle \tag{4}$$

3 Relating the spin bases

Now it just remains to figure out how we go back and forth between the two bases. We expect very generally for them to be related by linear combinations with various coefficients, i.e. a given state of definite s, m can be written as

$$|sms_1s_2\rangle = \sum_{m_1+m_2=m} C^{s_1\, s_2\, s}_{m_1\, m_2\, m} |s_1m_1\rangle|s_2m_2\rangle$$

where the C are numerical coefficients given by the usual completeness relation,

$$C^{s_1\, s_2\, s}_{m_1\, m_2\, m} \equiv (\langle s_1m_1|\langle s_2m_2|)|sms_1s_2\rangle$$

There is obviously a general procedure for working out these coefficients, but for the sake of definiteness, let's start by working out a simple example. The simplest case is that of the hydrogen atom where $s_1, s_2 = \frac{1}{2}$. Imagine further that we start with a state where both spins are up, i.e. $m_1 = m_2 = \frac{1}{2}$. We can just work out what happens when we act with S_z on this state:

$$S_z\left(|1/2\,1/2\rangle|1/2\,1/2\rangle\right) = \left(S_{1z}|1/2\,1/2\rangle\right)|1/2\,1/2\rangle + |1/2\,1/2\rangle S_{2z}\left(|1/2\,1/2\rangle\right) \tag{5}$$

$$= \frac{1}{2}\hbar|1/2\,1/2\rangle|1/2\,1/2\rangle + \frac{1}{2}\hbar|1/2\,1/2\rangle|1/2\,1/2\rangle \tag{6}$$

$$= \hbar|1/2\,1/2\rangle|1/2\,1/2\rangle \tag{7}$$

so we conclude $m = 1$ – which is no surprise, since we knew $m = m_1 + m_2$. What about finding s? Well, let's act with S^2. Recall we have $S^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2$, but we'd like to get this into a form that acts simply on the uncoupled representation, so we use

$$S_{ix} = \frac{1}{2}(S_{i+} + S_{i-}) \tag{8}$$

$$S_{iy} = \frac{i}{2}(S_{i-} - S_{i+}) \tag{9}$$

for $i = 1, 2$ to write

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}) + S_{1z}S_{2z}$$

Then we have very generally

$$\begin{aligned}\mathbf{S}_1 \cdot \mathbf{S}_2|s_1\,m_1\rangle|s_2\,m_2\rangle = &\frac{\hbar^2}{2}\sqrt{s_1(s_1+1) - m_1(m_1+1)}\sqrt{s_2(s_2+1) - m_2(m_2-1)}|s_1\,m_1+1\rangle|s_2\,m_2-1\rangle \\ &+\frac{\hbar^2}{2}\sqrt{s_1(s_1+1) - m_1(m_1-1)}\sqrt{s_2(s_2+1) - m_2(m_2+1)}|s_1\,m_1-1\rangle|s_2\,m_2+1\rangle \\ &+\hbar^2 m_1 m_2|s_1\,m_1\rangle|s_2 m_2\rangle\end{aligned}$$

This should make it even more clear that the states $|s_1\,m_1\rangle|s_2\,m_2\rangle$ are not eigenstates of S^2, since (part of) S^2 acting on such a state gives back a linear combination of states with other values of m_1, m_2.

3.1 $|1/2\,1/2\rangle|1/2\,1/2\rangle$

Putting this to work on our specific case of $|1/2\,1/2\rangle|1/2\,1/2\rangle$, we have

$$\mathbf{S}_1 \cdot \mathbf{S}_2|1/2\,1/2\rangle|1/2\,1/2\rangle = \frac{\hbar^2}{4}|1/2\,1/2\rangle|1/2\,1/2\rangle$$

and so

$$S^2|1/2\,1/2\rangle|1/2\,1/2\rangle = \hbar^2\left[2\times\frac{1}{2}(\frac{1}{2}+1)+\frac{1}{2}\right]|1/2\,1/2\rangle|1/2\,1/2\rangle = 2\hbar^2|1/2\,1/2\rangle|1/2\,1/2\rangle$$

from which we conclude $s=1$. Thus we find (suppressing the s_1, s_2 on the LHS for simplicity)

$$|1\,1\rangle = |1/2\,1/2\rangle|1/2\,1/2\rangle$$

i.e., this is an eigenstate of both sets of operators.

Now what is going on here? Well, we know that the combined spin in $\hat{z}$ is the sum of the individual spins in $\hat{z}$, which in this case was $m_1+m_2=1$. We also know that for a general spin state, $-s\le m\le s$, so it had to be the case that $s\ge 1$. Finally, it's physically sensible that the total spin of the two-particle system can't be larger than the sum of the total spins of each particles (though it could be *smaller*, if the spins partially cancel), which tells us $s\le 1$. This combination of facts uniquely fixes the answer, in agreement with our direct calculation. Although we cannot guess the answer for other spin states, there is a simple procedure for finding the rest.

Now we can find other eigenstates in the coupled representation by acting with S_- on the $|11\rangle$ state to lower the z component. This gives

$$S_-|11\rangle = \sqrt{2}\hbar|10\rangle = S_-|1/2\,1/2\rangle|1/2\,1/2\rangle = \left(S_{1-}|1/2\,1/2\rangle\right)|1/2\,1/2\rangle + |1/2\,1/2\rangle\left(S_{2-}|1/2\,1/2\rangle\right) \quad (10)$$

$$= \hbar\left(|1/2\,-1/2\rangle|1/2\,1/2\rangle + |1/2\,1/2\rangle|1/2\,-1/2\rangle\right) \quad (11)$$

where on the LHS we just used the definition of the lowering operator, and on the RHS we used the definition of the lowering operator in terms of lowering operators acting on each of the particles. Thus we conclude

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|1/2\,-1/2\rangle|1/2\,1/2\rangle + |1/2\,1/2\rangle|1/2\,-1/2\rangle\right)$$

Acting once more with S_-, we find

$$S_-|10\rangle = \sqrt{2}\hbar|1\,-1\rangle = \sqrt{2}\hbar|1/2\,-1/2\rangle|1/2\,-1/2\rangle$$

so

$$|1\,-1\rangle = |1/2\,-1/2\rangle|1/2\,-1/2\rangle$$

That exhausts our ability to find states by acting with S_-, but we are clearly not done; there are four eigenstates of s_i, m_i and we have found only three of s, m. Clearly the orthogonal combination we are missing is $\frac{1}{\sqrt{2}}\left(|1/2\,-1/2\rangle|1/2\,1/2\rangle - |1/2\,1/2\rangle|1/2\,-1/2\rangle\right)$,

and if we act with S^2 we find unsurprisingly that this is a state with $s = 0$. Thence the final eigenstate is

$$|00\rangle = \frac{1}{\sqrt{2}} \left(|1/2\ -1/2\rangle |1/2\ 1/2\rangle - |1/2\ 1/2\rangle |1/2\ -1/2\rangle \right)$$

and we are done. (Equivalently, you could have just checked by acting on $|00\rangle$ with raising and lowering operators; they both annihilate the state, and this fixes the coefficients.)

We have found all the nonzero coefficients; the nonzero ones are (in the $C^{s_1\, s_2\, s}_{m_1\, m_2\, m}$ notation)

$$C^{1/2\,1/2\,1}_{1/2\,1/2\,1} = 1 \qquad C^{1/2\,1/2\,1}_{-1/2\,-1/2\,-1} = 1 \tag{12}$$

$$C^{1/2\,1/2\,1}_{1/2\,-1/2\,0} = \frac{1}{\sqrt{2}} \qquad C^{1/2\,1/2\,1}_{-1/2\,1/2\,0} = \frac{1}{\sqrt{2}} \tag{13}$$

$$C^{1/2\,1/2\,1}_{1/2\,-1/2\,0} = -\frac{1}{\sqrt{2}} \qquad C^{1/2\,1/2\,1}_{-1/2\,1/2\,0} = \frac{1}{\sqrt{2}} \tag{14}$$

and the rest vanish.

This procedure generalizes naturally. In general, combining states of spins s_1 and s_2 can yield states of total spin ranging from $s_1 + s_2 \geq s \geq |s_1 - s_2|$. The intuition is that the spins can either add completely in parallel ($s = s_1 + s_2$) or completely antiparallel ($s = |s_1 - s_2|$) or anything in between. To *prove* this, note that the max value of m is

$$m_{max} = s_1 + s_2$$

so that

$$s_{max} = s_1 + s_2$$

as claimed. To find the minimum value, note in the uncoupled representation that there are $(2s_1 + 1)(2s_2 + 1)$ independent eigenstates corresponding to the possible values of m_1, m_2. This must also be the dimensionality of the space of eigenstates of S^2, S_z. So now we have a counting exercise, lowering s_{max} in unit steps until we get to $(2s_1 + 1)(2s_2 + 1)$ states. For a given s, the number of independent eigenstates is $2s + 1$, so s_{min} satisfies

$$\sum_{s=s_{min}}^{s_1+s_2} (2s + 1) = (2s_1 + 1)(2s_2 + 1)$$

from which one finds $s_{min} = |s_1 - s_2|$ as claimed.

Now for each state of spin s, m ranges as usual from $-s$ to s, and for each such state the coefficients relating these to the $|s_1\, m_1\rangle|s_2\, m_2\rangle$ can be obtained by using the raising and lowering operators.

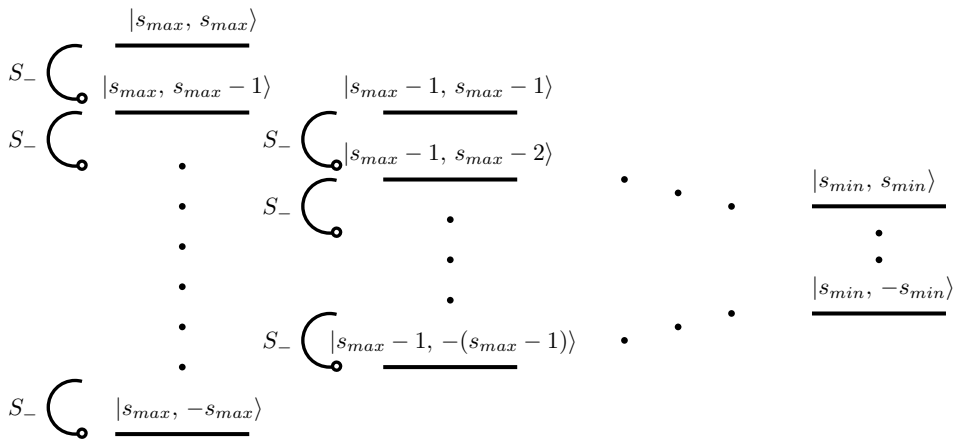

Figure 1: Sketch of how you can work out all the states of definite total spin $|s, m\rangle$ given individual spins s_1, s_2, starting with $s_{max} = s_1 + s_2, m = s_{max}$ and acting with the lowering operator S_- to obtain all states of lower m; then decreasing s in unit intervals and repeating the exercise for each s, terminating at $s_{min} = |s_1 - s_2|$. This gives a system of linear equations that can then be completely solved for the CG coefficients.

Of course, this procedure is clearly a mess for more general s_1, s_2 than we considered above. Thankfully, the world is a merciful place and someone has done this for you already once and for all. The coefficients $C^{s_1\, s_2\, s}_{m_1\, m_2\, m}$ are known as *Clebsch-Gordan coefficients*, and can be read off of tables. By convention the coefficients are taken to be real (they could have unphysical phases that we set to zero WLOG), so they allow you to read of $|sm\rangle$ in terms of $|s_1 m_1\rangle|s_2 m_2\rangle$ or visa versa.

We have focused on the case of two distinct particles, i.e. adding two spins, but the direct product of spins is associative, so if you want to work out the case of three or more spins you can just iterate the procedure for the product of two spins as long as necessary.

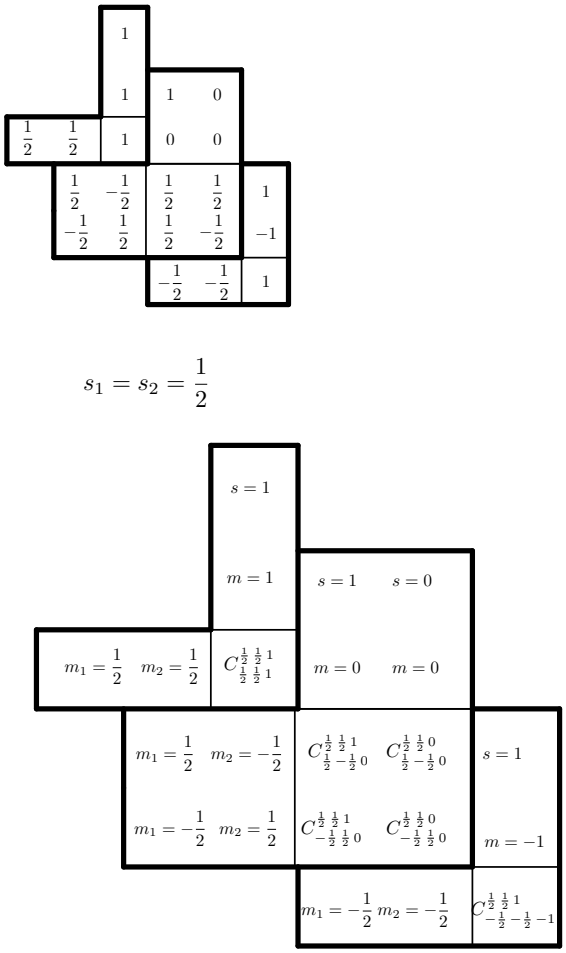

Figure 2: A CG coefficient table for the $s_1, s_2 = 1/2$ case. The top are the coefficients as they'd appear in a table; the bottom are the meanings. For the coefficients, square roots are implicit (with minus signs taken outside).

4 Other angular momenta

4.1 Addition of orbital angular momenta

We have worked through the case of adding spins, but adding orbital angular momentum works in the very same way; for two particles we define the total angular momentum

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L_2}$$

and compute total angular momentum ℓ and z-projection m in terms of the angular momenta ℓ_1, m_1 and ℓ_2, m_2 of the particles. The two relevant representations are $L_1^2, L_{1z}, L_2^2, L_{2z}$ and L^2, L_z, L_1^2, L_2^2, and everything goes through as before.

4.2 Addition of spin and angular momentum

Finally, consider the total angular momentum of a single particle, i.e. the sum of spin and orbital angular momentum:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

Note that $\mathbf{L}, \mathbf{S}$ commute because $\mathbf{L}$ is acting on the spatial part of the wavefunction and $\mathbf{S}$ is acting on the spin part, so

$$J^2 = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S} \tag{15}$$

$$J_z = L_z + S_z \tag{16}$$

We label the eigenvalues of J^2, J_z with j, m. Again, everything goes through as before: there are two possible representations, L^2, L_z, S^2, S_z or J^2, J_z, L^2, S^2, etc.

As a final tip, whether you are interested in adding spins, angular momentum, or both, you will invariably find the relationship

$$\mathbf{J}_1 \cdot \mathbf{J}_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right)$$

incredibly useful because states of the coupled representation are obviously eigenstates of the RHS, so you can simply read off the eigenvalues of $\mathbf{J}_1 \cdot \mathbf{J}_2$ acting on these states. That is to say,

$$\mathbf{J}_1 \cdot \mathbf{J}_2 \left|j, m, j_1, j_2\right\rangle = \frac{1}{2}\left(\hbar^2 j(j+1) - \hbar^2 j_1(j_1+1) - \hbar^2 j_2(j_2+1)\right)\left|j, m, j_1, j_2\right\rangle$$

5 Entanglement

I want to close by returning to an interesting point that we mentioned last lecture: although the two-particle states in the uncoupled representation all have the form $|1\rangle \otimes |2\rangle$ (i.e. they factorize into a product of the states of particle one and particle two), the states of definite total spin in the coupled representation cannot always be written this way: they are linear combinations of products that don't factorize. This is the simplest example of *entanglement*, which is one of the truly original properties of quantum mechanics.

That is to say, while the state

$$|1\,1\rangle = |{}^1\!/_2\,{}^1\!/_2\rangle|{}^1\!/_2\,{}^1\!/_2\rangle$$

is just the product of two single-particle states, the states

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|{}^1\!/_2\,{}^{-1}\!/_2\rangle|{}^1\!/_2\,{}^1\!/_2\rangle + |{}^1\!/_2\,{}^1\!/_2\rangle|{}^1\!/_2\,{}^{-1}\!/_2\rangle\right) \tag{17}$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|{}^1\!/_2\,{}^{-1}\!/_2\rangle|{}^1\!/_2\,{}^1\!/_2\rangle - |{}^1\!/_2\,{}^1\!/_2\rangle|{}^1\!/_2\,{}^{-1}\!/_2\rangle\right) \tag{18}$$

are *not* simply the product of two single-particle states. Such states are said to be *entangled*. The second example ($|00\rangle$) is perhaps the most famous of all entanged states, and is known as the EPR pair, for Einstein, Podolsky, and Rosen. You will explore all of the remarkable properties of entangled particles in 115C, though I will also post some bonus notes for your amusement.