We said that the rotation operator

$$R(\vec{\theta}) = \exp\left(-i\frac{\vec{\theta}\vec{L}}{\hbar}\right)$$

i.e angular momentum is the generator of rotations. Let's try to justify it a bit better.

Under rotations $\vec{r} \to \vec{r}\,' = R\vec{r}$

$$|\psi\rangle \to |\psi'\rangle$$

In terms of wavefunctions $\boxed{\psi'(\vec{r}\,') = \psi(\vec{r})}$

If you are not convinced of this, think in terms of probability densities. You rotate the system, but the probability density at the new position must be the same as it was at the old position, thus

$$|\psi(\vec{r})|^2 = |\psi'(\vec{r}\,')|^2$$

Then $\psi'(\vec{r}\,') = \psi(\vec{r}) = \psi(R^{-1}\vec{r}\,')$

relabeling $\vec{r}\,'$ as $\vec{r}$

$$\boxed{\psi'(\vec{r}) = \psi(R^{-1}\vec{r})}$$

This is the new wavefunction!

Now consider rotation around z by θ

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For small θ, $R(\theta) \simeq \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$R^{-1}(\theta) = \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The new wavefunction is

$$\psi(R^{-1}\vec{r}) = \psi(x + y\theta \quad y - x\theta \quad z)$$

Expand in Taylor Series

$$\psi(R^{-1}\vec{r}) = \psi(x,y,z) + y\theta\frac{\partial\psi}{\partial x} - x\theta\frac{\partial\psi}{\partial y}$$

$$\psi(R^{-1}\vec{r}) = \psi(\vec{r}) - i\theta\underbrace{\left(iy\frac{\partial}{\partial x} - ix\frac{\partial}{\partial y}\right)}_{=\frac{1}{\hbar}L_z}\psi(\vec{r})$$

$$\boxed{\psi(R^{-1}\vec{r}) = \psi'(\vec{r}) = \left(1 - \frac{i\theta L_z}{\hbar}\right)\psi(\vec{r})}$$

For small rotation by θ around z-axis –

We had that for rotation by θ around z axis

$$R(\theta) = \exp\left(-\frac{i\theta L_z}{\hbar}\right)$$

and for small θ

$$\boxed{R(\theta) = 1 - \frac{i\theta L_z}{\hbar}}$$