(Brief) Solutions to Practice Final Exam, Physics 115B

This exam is closed book, closed notes, closed calculators/phones. Please show your work for full credit. You may make free use of anything on the "Useful Formulae" page. There are seven problems.

You have 180 minutes. You can do this!

Useful formulae

Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$ w/ Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V$ (any space)

Harmonic oscillator (1d): For $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, raising/lowering operators $a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p})$, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$, $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$, $[a_-, a_+] = 1$, $\hat{H} = \hbar\omega(a_+a_- + 1/2)$, $E_n = \hbar\omega(n + 1/2)$, $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\psi_{n-1}$, $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$.

QM in 3D: Position operator $\vec{x} = (x, y, z)$ and momentum operator $\vec{p} = (p_x, p_y, p_z)$ in Cartesian coords. Position space $p_x = -i\hbar\frac{\partial}{\partial x}, p_y = -i\hbar\frac{\partial}{\partial y}, p_z = -i\hbar\frac{\partial}{\partial z}$, so $\vec{p} = -i\hbar\nabla$ and $H = -\frac{\hbar^2}{2m}\nabla^2 + V$. Commutators $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$, all other commutators of x, y, z, p_x, p_y, p_z are zero.

Spherically symmetric potentials: $V(\vec{r}) = V(r)$, eigenstates $\psi_{n,\ell,m} = R_{n,\ell}(r)Y_\ell^m(\theta, \phi)$, radial momentum $p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$ and $p_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$.

Hydrogen atom: $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$, energies $E_n = -\frac{\hbar^2}{2ma_0^2n^2} = -\mathcal{R}/n^2$, Bohr radius $a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2}$, ground state wavefunction $\psi_{1,0,0}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$, "circular orbit" wavefunctions $\psi_{n,n-1,m} = \frac{1}{\sqrt{(2n)!}}\left(\frac{2}{na_0}\right)^{3/2}\left(\frac{2r}{na_0}\right)^{n-1} e^{-r/na_0}\, Y_{n-1}^m(\theta, \phi)$.

Angular Momentum & Spin Operators: $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$, analogous relations for S_x, S_y, S_z, and total angular momentum $\vec{J} = \vec{L} + \vec{S}$.
$L^2|\ell, m\rangle = \hbar^2\ell(\ell+1)|\ell, m\rangle$, $L_z|\ell, m\rangle = \hbar m|\ell, m\rangle$, analogous expressions for S^2, S_z acting on $|s, m\rangle$ and J^2, J_z acting on $|j, m\rangle$. Raising and lowering operators $L_\pm = L_x \pm iL_y$ for eigenstates of L^2, L_z, with $L_+|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|\ell, m+1\rangle$ and $L_-|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|\ell, m-1\rangle$. In terms of these operators, $L_x = \frac{1}{2}(L_+ + L_-)$ and $L_y = \frac{1}{2i}(L_+ - L_-)$; analogous expressions for $S_\pm$ and $J_\pm$.

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Representation of spin-1/2: Basis $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, operators $\vec{S} = (\hbar/2)\vec{\sigma}$

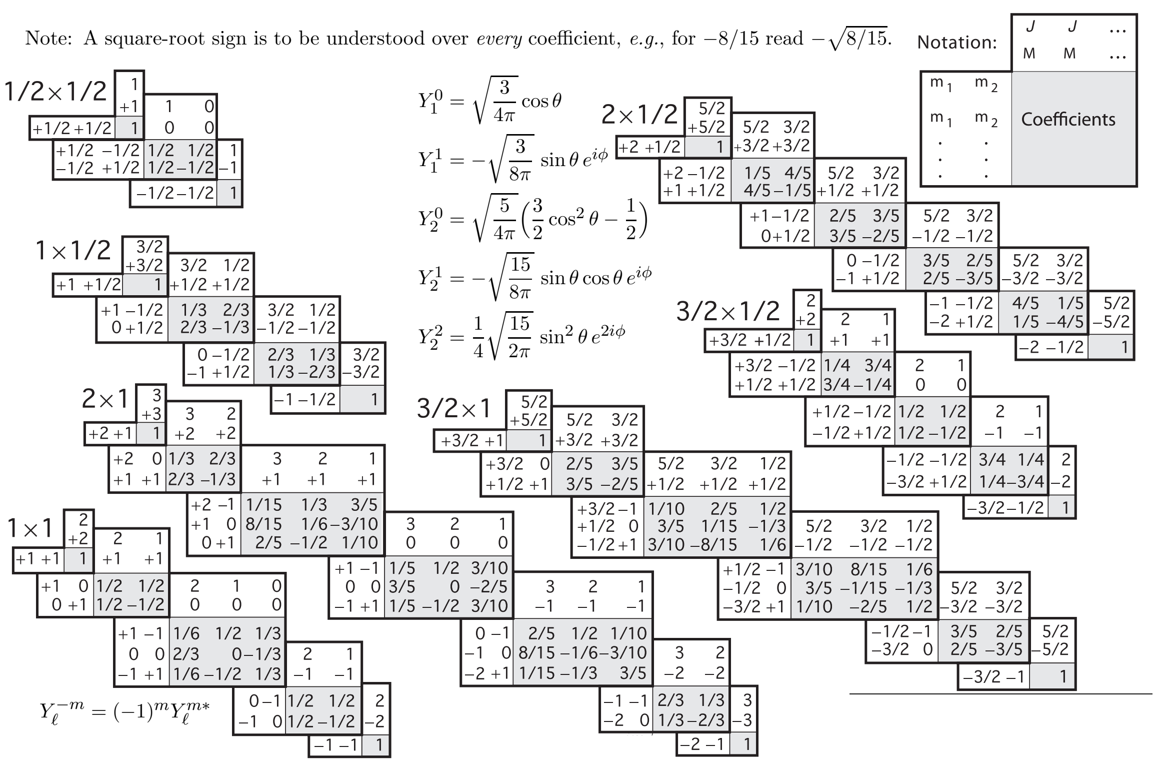
Coupling to electromagnetism: For a particle of charge q, $H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\varphi$, where $\vec{p}$ is the vector of momentum operators (i.e. $\vec{p} = (p_x, p_y, p_z)$), $\vec{A}$ is the vector potential, φ is the scalar potential, in terms of which the electric and magnetic fields are $\vec{E} = -\nabla\varphi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla \times \vec{A}$. Gauge transformations $\varphi' \equiv \varphi - \partial\Lambda/\partial t, \vec{A}' \equiv \vec{A} + \nabla\Lambda$ leave the physics unchanged.

Free electron gas: Box of volume $V = L_x L_y L_z$. States labeled by wavenumbers $\vec{k} = (n_x\pi/L_x, n_y\pi/L_y, n_z\pi/L_z)$, each occupies k-space volume π^3/V. Electrons in ground state fill out states to Fermi radius $k_F = (3\rho\pi^2)^{1/3}$ in terms of free electron density $\rho = Nd/V$. Energy of the Fermi surface $E_F = \frac{\hbar^2 k_F^2}{2m}$, total energy $E_{tot} = \frac{\hbar^2 k_F^5 V}{10\pi^2 m} = \frac{\hbar^2 (3\pi^2 Nd)^{5/3}}{10\pi^2 m} V^{-2/3}$. Work $dW = PdV$ done by pressure implies degeneracy pressure $P = \frac{2}{3}\frac{E_{tot}}{V}$.

Symmetries and Conservation Laws: Active transformation: act on states, $|\psi\rangle \to T|\psi\rangle$, operators fixed. Passive transformation: act on operators, $\mathcal{O} \to T^\dagger \mathcal{O} T$, states fixed. Symmetry if $[H, \mathcal{O}] = 0$, implies conservation laws $\frac{d}{dt}\langle\mathcal{O}\rangle = 0$.

Transformations: Transformations unitary, $T^\dagger T = 1$. Translation operator $T(a)|x\rangle = |x+a\rangle$, $T(a)\psi(x) = \psi(x-a)$, generated by momentum, $T(a) = \exp[-ia\hat{p}/\hbar]$. Parity operator $\Pi|x\rangle = |-x\rangle$, $\Pi\psi(x) = \psi(-x)$. Rotation operator $R(\hat{n},\theta)$ by θ around axis $\hat{n}$, generated by angular momentum, $R(\hat{n},\theta) = \exp\left[-i\theta\hat{n}\cdot\vec{L}/\hbar\right]$. Time translation operator $U(t)$, generated by Hamiltonian, $U(t) = \exp[-iHt/\hbar]$.

Clebsch-Gordan Coefficients



1. Is each of the following statements true or false? If true, simply write "true". If false, you must *briefly* (at most one sentence) explain why the statement is false.

   (a) The first excited state of the 3d harmonic oscillator is threefold degenerate.

   (b) The observable corresponding to radial momentum is $\hat{r} \cdot \vec{p}$.

   (c) The "circular" orbits of the Hydrogen atom are the ones with the smallest possible angular momentum.

   (d) The energy of electromagnetic radiation emitted when an electron in the excited state of hydrogen with principle quantum number n transitions to the ground state is $E = -\mathcal{R}\left(\frac{1}{n^2} - 1\right)$.

   (e) $[L_x, L_z] = i\hbar L_y$

   (f) The eigenstates of orbital angular momentum are the generalized Legendre polynomials P_ℓ^m.

   (g) The total angular momentum of a particle is given by the addition of its intrinsic spin and orbital angular momentum.

   (h) Electrons can have different values of total intrinsic spin.

   (i) $|1, -1\rangle = |1/2, -1/2\rangle\, |1/2, 1/2\rangle$

   (j) $T(a)T(b) = T(a + b)$ for the spatial translation operator T.

   (k) $\hat{x} \to \hat{x}$ under parity transformations.

   (l) The time translation operator is hermitian.

   Solutions:

   (a) T

   (b) F, not hermitian.

   (c) F, largest possible.

   (d) T

   (e) F, minus sign.

   (f) F, spherical harmonics.

   (g) T

   (h) F, all electrons are $s = 1/2$.

   (i) F, both spins down.

   (j) T

   (k) F, $\hat{x} \to -\hat{x}$.

   (l) F, unitarity but not hermitian.

2. Provide a short written answer (either an equation or 1-2 sentences) to each of the following questions.

   (a) Suppose you measure S_z for a spin-1/2 particle and find $+\hbar/2$. You then measure S_x, and then measure S_z once again. What values could you obtain for the second measurement of S_z, and with what probabilities?

   (b) Given three indistinguishable bosons, one in each in the single-particle states ψ_a, ψ_b, and ψ_c, construct the appropriate normalized three-particle wavefunction.

   (c) A particle of total spin $s_1 = 1/2$ and a particle of total spin $s_2 = 1$ are coupled together via a Hamiltonian of the form $H = \epsilon \vec{S}_1 \cdot \vec{S}_2$. What are the eigenvalues of the Hamiltonian?

   (d) Consider a hydrogen atom in the orbital state $\psi_{2,1,1}$, with both proton and electron spins oriented down in the $\hat{z}$ direction. If you measured the total angular momentum J^2 of the hydrogen atom (including the spins of the electron and the proton), what are the possible outcomes, and with what probabilities?

   (e) Consider two particles in the spin state $|1,0\rangle\,|1,0\rangle$. If we measured the magnitude of the total spin S^2 for the two-particle system, what values could we get, and with what probabilities?

Solutions:

(a) $+\hbar/2$ *with probability 1/2 and* $-\hbar/2$ *with probability 1/2.*

(b)

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \frac{1}{\sqrt{6}} \left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2)\psi_c(\vec{r}_3) + \psi_a(\vec{r}_1)\psi_c(\vec{r}_2)\psi_b(\vec{r}_3) + \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\psi_c(\vec{r}_3)\right. \quad (1)$$

$$\left. + \psi_b(\vec{r}_1)\psi_c(\vec{r}_2)\psi_a(\vec{r}_3) + \psi_c(\vec{r}_1)\psi_b(\vec{r}_2)\psi_a(\vec{r}_3) + \psi_c(\vec{r}_1)\psi_a(\vec{r}_2)\psi_b(\vec{r}_3)\right] \quad (2)$$

(c) We have $H = \frac{\epsilon}{2}(S^2 - S_1^2 - S_2^2)$ *so the eigenvalues are* $\frac{\hbar^2\epsilon}{2}[s(s+1) - 11/4]$. *The possible values of* s *are* $s = 1/2, 3/2$ *so the eigenvalues are* $-\epsilon\hbar^2, +\epsilon\hbar^2/2$.

(d) The addition of electron and proton spins gives

$$|1/2, -1/2\rangle\,|1/2, -1/2\rangle = |1, -1\rangle$$

Adding this to the orbital angular momentum state $|1,1\rangle$ *gives*

$$|1,1\rangle\,|1,-1\rangle = \frac{1}{\sqrt{6}}\,|2,0\rangle + \frac{1}{\sqrt{2}}\,|1,0\rangle + \frac{1}{\sqrt{3}}\,|0,0\rangle$$

so the possible outcomes are $j = 2$ *with probabilitt 1/6,* $j = 1$ *with probability 1/2, and* $j = 0$ *with probability 1/3.*

(e) We have

$$|1,0\rangle\,|1,0\rangle = \sqrt{\frac{2}{3}}\,|2,0\rangle - \sqrt{\frac{1}{3}}\,|0,0\rangle$$

so the outcomes are $s = 2$ *with probability 2/3 and* $s = 0$ *with probability 1/3.*

3. This problem involves the decay of an unstable particle C to particles A and B, in which total angular momentum is conserved. In the rest frame of C, the total angular momentum $\vec{J} = \vec{S}_C$ is just the spin of the particle C. After the decay, the total angular momentum consists of three terms,

$$\vec{J} = \vec{S}_A + \vec{S}_B + \vec{L}$$

where $\vec{S}_A$ is the spin of particle A, $\vec{S}_B$ is the spin of particle B, and $\vec{L}$ is the orbital angular momentum between A and B. Conservation of angular momentum in this decay means that if the initial state is an eigenstate of J^2 and J_z, then the final state is also an eigenstate with the same eigenvalues.

(a) Consider the case where C is a spin-0 particle and A, B are both spin-1/2 particles ($s_A = s_B = 1/2$). What values of the orbital angular momentum ℓ are consistent with angular momentum conservation?

(b) Repeat the above problem, but now where C is a spin-3/2 particle, A is a spin-1/2 particle, and B is a spin-1 particle.

(c) There are certain processes for which a two-body decay is forbidden. Explain why a neutron n cannot decay to a proton p and an electron e^- (all spin-1/2 fermions), despite this being consistent with energy and charge conservation.

(d) A mystery particle C of unknown spin s_C is polarized such that $m_C = +s_C$. It decays into particles A & B, where $s_A = 1/2$ and $s_B = 0$, but the relative orbital angular momentum ℓ is unknown (for simplicity, you may assume the decay gives a single, but unknown, value of ℓ). After the decay, the z component m_A of the spin of particle A is measured, and is found to have probabilities

$$P(m_A = 1/2) = 1/5 \qquad P(m_A = -1/2) = 4/5$$

What is s_C, and what is ℓ? *Hint: you may wish to consider the action of J_+ on both initial and final states.*

Parts (a), (b), and (c) are identical to Problem 4 of Homework 6. For part (d) we have:
we can make our lives simpler at the start by noting that the combined spin states are just $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$; the spin of B is zero. Then remembering also that $m_C = m_A + m_B + m_\ell$, we have

$$|s_C, s_C\rangle = \alpha\, |1/2, 1/2\rangle\, |\ell, s_C - 1/2\rangle + \beta\, |1/2, -1/2\rangle\, |\ell, s_C + 1/2\rangle$$

Then acting on both sides with $J_+ = S_+ + L_+$ gives

$$0 = \hbar\beta\, |1/2, 1/2\rangle\, |\ell, s_C + 1/2\rangle + \hbar c_1 \alpha\, |1/2, 1/2\rangle\, |\ell, s_C + 1/2\rangle + \hbar c_2 \beta\, |1/2, -1/2\rangle\, |\ell, s_C + 3/2\rangle$$

with $c_1 = \sqrt{\ell(\ell+1) - (s_C - 1/2)(s_C + 1/2)}$ *and* $c_2 = \sqrt{\ell(\ell+1) - (s_C + 1/2)(s_C + 3/2)}$. *This implies* $\beta = -c_1\alpha$ *and* $c_2\beta = 0$. *The only nontrivial solution to these is* $c_2 = 0$, *which implies*

$$\ell(\ell+1) = (s_C + 1/2)(s_C + 3/2)$$

which is nominally satisfied for either $\ell = -s_C - 3/2$ *or* $\ell = s_C + 1/2$, *but only the latter is consistent with* $\ell > 0$ *and* $s_C > 0$. *This means our state is*

$$|s_C, s_C\rangle = \alpha\,|1/2, 1/2\rangle\,|s_C + 1/2, s_C - 1/2\rangle + \beta\,|1/2, -1/2\rangle\,|s_C + 1/2, s_C + 1/2\rangle$$

The only entries in the C-G table consistent with this decomposition and the stated outcomes of measurements of S_z *for particle* A *are those for which* $s_C = 3/2$, *i.e.*

$$|3/2, 3/2\rangle = \sqrt{\frac{4}{5}}\,|1/2, -1/2\rangle\,|2, 2\rangle - \sqrt{\frac{1}{5}}\,|1/2, 1/2\rangle\,|2, 1\rangle$$

so we conclude $s_C = 3/2$ *and* $\ell = 2$.

4. Ignoring electron-electron repulsion, construct the ground state of Lithium ($Z = 3$). Start with a spatial wave function, remembering that only two electrons can occupy the hydrogenic ground state; the third goes to $\psi_{2,0,0}$. What is the energy of this state? Now tack on the spin, and antisymmetrize. What's the degeneracy of the ground state?

   Solution: This is the same as Problem 3 of Homework 7.

5. In this problem, you will calculate the effect of the exchange interaction in an infinite square well. Consider two noninteracting particles of mass m. Recall the infinite square well eigenstates are

$$\psi_j(x) = \sqrt{\frac{2}{a}} \sin \frac{j\pi x}{a}$$

where $j = 1, 2, 3, \ldots$ and $x \in [0, a]$. Let the particles be in states ψ_j and ψ_k with $j \neq k$. Then

$$\left\langle (\Delta x)^2 \right\rangle_d = \langle \psi_j | \, x^2 \, | \psi_j \rangle + \langle \psi_k | \, x^2 \, | \psi_k \rangle - 2 \, \langle \psi_j | \, x \, | \psi_j \rangle \, \langle \psi_k | \, x \, | \psi_k \rangle$$

$$\left\langle (\Delta x)^2 \right\rangle_\pm = \left\langle (\Delta x)^2 \right\rangle_d \mp 2 | \, \langle \psi_j | \, x \, | \psi_k \rangle \, |^2,$$

where the subscript d refers to the distinguishable particle state and $\pm$ refers to the symmetric or antisymmetric state. Note the following useful integrals:

$$\int_0^a dx \, x \sin^2(j\pi x/a) = a^2/4$$

$$\int_0^a dx \, x^2 \sin^2(j\pi x/a) = \frac{a^3}{12} \left(2 - \frac{3}{\pi^2 j^2} \right)$$

$$\int_0^a dx \, x \sin(j\pi x/a) \sin(k\pi x/a) = \frac{2a^2 jk \left[(-1)^{j+k} - 1 \right]}{\pi^2 (j^2 - k^2)^2}$$

Calculate $\left\langle (\Delta x)^2 \right\rangle$ if the particles are

(a) distinguishable

(b) in a symmetric spatial wave function, with $j + k$ even.

(c) in a symmetric spatial wave function, with $j + k$ odd.

(d) in an antisymmetric spatial wave function, with $j + k$ even.

(e) in an antisymmetric spatial wave function, with $j + k$ odd.

Solution: This is the same as Problem 3 of Homework 8.

6. Consider two particles of mass m_1 and m_2 (in one dimension) that interact via a potential that depends only on the distance between the particles, $V(|x_1 - x_2|)$, so that the Hamiltonian is

$$H = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(|x_1 - x_2|).$$

Acting on a two-particle wave function, the translation operator would be

$$\hat{T}(a)\psi(x_1, x_2) = \psi(x_1 - a, x_2 - a)$$

(a) Show that the translation operator can be written

$$\hat{T}(a) = \exp\left[-\frac{ia}{\hbar}\hat{P}\right]$$

where $\hat{P} = \hat{p}_1 + \hat{p}_2$ is the total momentum.

(b) Show that the total momentum is conserved for this system.

Solutions:

(a) Various possible ways to do this. One is to note that for a single particle

$$T(a)\psi(x) = \psi(x - a)$$

for which the translation operator can be written as

$$T(a) = \exp\left[-iap/\hbar\right]$$

As confirmation,

$$\begin{aligned} T(a)\psi(x) &= \left(1 - \frac{ia}{\hbar}p + \left(\frac{-ia}{\hbar}\right)^2 p^2 + \ldots\right)\psi(x) \\ &= \left(1 - a\frac{d}{dx} + a^2\frac{d^2}{dx^2} + \ldots\right)\psi(x) \\ &= \psi(x) - \frac{d\psi}{dx}a + \frac{d^2\psi}{dx^2}a^2 + \ldots \\ &= \psi(x - a) \end{aligned}$$

Now turning to two particles, since the operators p_1, p_2 commute, it follows that the product of two distinct single-particle translations

$$T(a) = \exp\left[-\frac{ia}{\hbar}p_1\right]\exp\left[-\frac{ia}{\hbar}p_2\right] = \exp\left[-\frac{ia}{\hbar}P\right]$$

satisfies

$$T(a)\psi(x_1, x_2) = \psi(x_1 - a, x_2 - a)$$

(b) *One line of argumentation would be to re-write the Lagrangian in terms of relative and center-of-mass coordinates. Since the potential depends only on the relative coordinate and is independent of the center-of-mass coordinate, it is invariant under center-of-mass translations and so total momentum is conserved.*

Another option would be to show that $[P, V] = 0$ *directly, i.e.*

$$\begin{aligned}
\frac{i}{\hbar}[P, V(|x_1 - x_2|)] &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) V(|x_1 - x_2|) - V(|x_1 - x_2|)\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) \\
&= V'(|x_1 - x_2|)\frac{(x_1 - x_2)}{|x_1 - x_2|} - V'(|x_1 - x_2|)\frac{(x_1 - x_2)}{|x_1 - x_2|} \\
&= 0
\end{aligned}$$

Thus $[P, H] = [P, V] = 0$ *and so total momentum is conserved.*

7. At time $t = 0$, an electron in a hydrogen atom is in the state

$$\psi(\vec{r}, 0) = A\left[3i\psi_{1,0,0}(\vec{r}) - 4\psi_{2,1,1}(\vec{r}) - i\psi_{2,1,0}(\vec{r}) + \sqrt{10}\psi_{2,1,-1}(\vec{r})\right]$$

where $\psi_{n,\ell,m}$ are the properly normalized energy eigenstates.

(a) Determine A.
(b) What is the wavefunction at time t, i.e. $\psi(\vec{r}, t)$?
(c) What is the expectation value $\langle E \rangle$ at $t = 0$? (In terms of E_1.)
(d) What is the expectation value $\langle L^2 \rangle$ at $t = 0$?
(e) What is the expectation value $\langle L_z \rangle$ at $t = 0$?
(f) Which of $\langle E \rangle, \langle L_z \rangle, \langle L^2 \rangle$, and $\langle \vec{r} \rangle$ vary with time in this state?
(g) Suppose that a measurement of L_z at $t = 0$ yields $\hbar$. After this measurement, what is the properly normalized wavefunction $\psi(\vec{r}, t)$?

Solutions:

(a) *We have*

$$|\psi|^2 = |A|^2 (9 + 16 + 1 + 10) = 36|A|^2 \to A = \frac{1}{6}$$

(b) *It's*

$$\psi(\vec{r}, t) = \frac{1}{6}\left[3ie^{-iE_1t/\hbar}\psi_{1,0,0}(\vec{r}) - 4e^{-iE_2t/\hbar}\psi_{2,1,1}(\vec{r}) - ie^{-iE_2t/\hbar}\psi_{2,1,0}(\vec{r}) + \sqrt{10}e^{-iE_2t/\hbar}\psi_{2,1,-1}(\vec{r})\right]$$

where E_1 *is the ground state energy and* $E_2 = E_1/4$.

(c) *We have*

$$\begin{aligned}\langle H\rangle &= \frac{1}{36}\left(9E_1 + 16E_2 + E_2 + 10E_2\right)\\ &= \frac{E_1}{36}\left(9 + 27/4\right) = \frac{7E_1}{16}\end{aligned}$$

(d) *We have*

$$\begin{aligned}\langle L^2\rangle &= \frac{\hbar^2}{36}\left(9\cdot 0 + 16\cdot 2 + 1\cdot 2 + 10\cdot 2\right)\\ &= \frac{3\hbar^2}{2}\end{aligned}$$

(e) *We have*

$$\begin{aligned}\langle L_z\rangle &= \frac{\hbar}{36}\left(9\cdot 0 + 16\cdot 1 + 1\cdot 0 + 10\cdot(-1)\right)\\ &= \frac{\hbar}{6}\end{aligned}$$

(f) $\langle E\rangle$ *is constant because energy is conserved.* $\langle L_z\rangle$ *and* $\langle L^2\rangle$ *are constant because angular momentum is conserved.* $\langle \vec{r}\rangle$ *varies in time because there are nonzero matrix elements between different components of the state with different energies.*

(g) *The outcome at* $t = 0$ *tells us we have been projected into the components of the state with* $m = 1$*. Thus we end up in* $\psi_{2,1,1}$ *and the state as a function of time is*

$$\psi(\vec{r}, t) = e^{-iE_2 t/\hbar}\psi_{2,1,1}(\vec{r})$$

CONGRATULATIONS! You've reached the end of the exam.