Transformation of the form $U(x) = e^{-i\alpha \cdot k}$

- $\alpha$ real, $k$ set of operators - Generators of transformation
  - $U^T U = 1$; $U = 1 - i\alpha \cdot k + \ldots$; $U^{-1} = U^T$
  - $U^T U = (1 + i\alpha \cdot k^T \ldots)(1 - i\alpha \cdot k + \ldots)$
  - $U^T U = 1 + i\alpha \cdot (k^T \cdot k) + O(\alpha^2) + \ldots$

$K^T = k$ hermitian, observable

We have seen transitions $\hat{k} = \hat{P}/\hbar$

- rotations $\hat{r} = \hat{J}/\hbar$
- isospin $\hat{r}^i = \hat{\sigma}/2$

Good symmetry: $U^T H U = H$

(1) Transitions $a \rightarrow b$

- $A_{ab} = \langle b | \hat{o} | a \rangle$
  - strength of transition $\propto |A_{ab}|^2$

$\hat{o}$ contains details of interaction, function of $H$ $\{U \hat{o} \} = 0$

Consider:
- $|a^i\rangle = U |a\rangle$
- $|b^i\rangle = U |b\rangle$

$A = \langle b | U^T \hat{o} U | a \rangle = \langle b^i | U | a^i \rangle = A_{ab}$

$\Rightarrow$ same amplitude for transitions between states related by symmetry operation

(2) $[k; H] = 0$

Simple to show
Consider infinitesimal transformation
\[ U = 1 - \alpha_x K = 1 - i \alpha_x k_i \quad (x \text{ small}) \]
\[ U^* H U = H \]
\[ (1 + i \alpha_x k_i^+)(1 - i \alpha_x k_i) = H \]
To first order in \( \alpha_x \), each using \( k_i^+ = k_i \)
\[ H + i \alpha_x (k_i H - H k_i) = H \]
\[ \text{must be zero} \Rightarrow [k_i; H] = 0 \]

3. Conservation of quantum numbers

\[ k_i |e\rangle = k_e |e\rangle \]
\[ k_i |b\rangle = k_b |b\rangle \]
Since \( [k; \theta] = 0 \)
\[ \langle b | [k; \theta] | e\rangle = 0 \]
\[ \langle b | k_0 - k l | e\rangle = 0 \quad (\text{remember } k^+ = k) \]
\[ (k_b - k_e) \langle b l | 0 | e\rangle = 0 \]
either \( \langle b l | 0 | e\rangle = 0 \quad \text{or} \quad k_b = k_e \)

4. Degeneracy

Take \( |e\rangle \) to be eigenstate of \( H \)
\[ H |e\rangle = \Omega E_a |e\rangle \quad |b\rangle = U |e\rangle \quad (\text{Assume } |a\rangle \neq |b\rangle, \text{ otherwise trivial}) \]
\[ E_a = \langle a | H | e\rangle = \langle a | U^* H U | e\rangle = \langle b | H | b\rangle = E_b \]
\[ E_a = E_b \]

States related by symmetry transform are degenerate.
Continuous symmetries lead to additive conservation laws

\[ AB \rightarrow a \cdot b \cdot c \]

Initial state: \[ |P_a \rangle \otimes |P_B \rangle \]
Final state: \[ |\tilde{P}_a \rangle \otimes |\tilde{P}_B \rangle \]

Translation

\[ U(\alpha) |i\rangle = e^{-i \alpha \frac{\hat{P}_a}{\hbar}} |P_a \rangle \otimes e^{-i \alpha \frac{\hat{P}_B}{\hbar}} |P_B \rangle \]

\[ U(\alpha) |i\rangle = e^{-i \alpha \frac{\hat{P}_a}{\hbar}} |P_a \rangle \otimes |P_B \rangle \]

where \[ \hat{P}_C = \frac{\hat{P}_A + \hat{P}_B}{2} \]

Similarly, for \[ U(\alpha) |f\rangle = e^{-i \alpha \frac{\hat{P}_F}{\hbar}} |f\rangle \]

\[ \langle f|01i \rangle = \langle f|0^+0u1i \rangle = e^{i \alpha (\tilde{P}_F - \tilde{P}_C)} \langle f|01i \rangle \]

Either \[ \langle f|01i \rangle = 0 \quad \text{or} \quad \tilde{P}_F = \tilde{P}_C \]

Discrete symmetries \( \rightarrow \) multiplicative conservation laws

E.g. Penta - Not continuous, no generator

We already saw, possible eigenvalues \( \pm 1 \)

\[ P(a) = P_b |a\rangle = P_0 |a\rangle \quad P(b) = P_b |b\rangle \]

\[ \langle b|01a \rangle = \langle b|0^+0a \rangle = P_0 P_b \langle b|01a \rangle \]

Either \[ P_0 P_b = 1 \quad \text{or} \quad \langle b|01a \rangle = 0 \]

\[ P_0 = 1 \quad P_0 = -1 \]

\[ P_b = 1 \quad P_b = -1 \]

These are the possibilities