PHYSICS 115B SPRING 2022

MIDTERM

1

(a) In Cartesian $\hat{r}\cdot\vec{p} = \frac{x_i}{r} p_i$

where p_i is the momentum operation in the ith direction

$(\hat{r}\vec{p})^{\dagger} = \vec{p}\,\hat{r} = p_i \frac{x_i}{r} \neq \frac{x_i p_i}{r}$ since $[p_i, x_i] \neq 0$

(b) $n=3$ $\quad l=0 \quad m=0$

$l=1 \quad m=0,\pm1$

$l=2 \quad m=0,\pm1,\pm2$

$\boxed{\#\text{ of states} = 9}$

(c) $\boxed{\text{False}}$ - The electron is pointlike

(d) $(S_z)_{ij} = \langle u_i | S_z | u_j \rangle = \begin{cases} 0 & \text{if } i\neq j \\ \frac{1}{2}\hbar & \text{if } i=1 \\ -\frac{1}{2}\hbar & \text{if } i=2 \end{cases}$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

2

page 2

(a) $|\psi\rangle = a|1\ 1\rangle + b|1\ 0\rangle + c|1\ -1\rangle$

$$L_x = \tfrac{1}{2}L_+ + \tfrac{1}{2}L_-$$

$$L_x|\psi\rangle = \frac{1}{2}L_+|\psi\rangle + \frac{1}{2}L_-|\psi\rangle$$

Need to calculate $L_\pm|1\ m\rangle$

$L_+|1\ 1\rangle = 0 \quad L_+|1\ 0\rangle = \sqrt{2}\hbar|1\ 1\rangle \quad L_+|1\ -1\rangle = \sqrt{2}\hbar|1\ 0\rangle$

$L_-|1\ 1\rangle = \sqrt{2}\hbar|1\ 0\rangle \quad L_-|1\ 0\rangle = \sqrt{2}\hbar|1\ -1\rangle \quad L_-|1\ -1\rangle = 0$

Therefore

$$L_x|\psi\rangle = \frac{\sqrt{2}}{2}\hbar\left[a|1\ 0\rangle + b|1\ 1\rangle + b|1\ -1\rangle + c|1\ 0\rangle\right]$$

$$L_x|\psi\rangle = \frac{\sqrt{2}}{2}\hbar\left[b|1\ 1\rangle + (a+c)|1\ 0\rangle + b|1\ -1\rangle\right]$$

We want $L_x|\psi\rangle = k|\psi\rangle$ with $k=0$

This gives $b=0$ and $a=-c$

$$\boxed{\Rightarrow\ |\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\ 1\rangle - |1\ -1\rangle\right)}$$

(b) The other eigenstates are obtained by solving $ka = kc = b$ and $kb = a+c$

$k(a+c) = 2b$

$(a+c) = kb$ $\Rightarrow$ $k = \pm\sqrt{2}$

where I dropped common factors of $\hbar$

page 3

Therfore the other two states, up to a normalization constant are

$$|\psi\rangle = |1\ 1\rangle \pm \sqrt{2}\,|1\ 0\rangle + |1\ {-1}\rangle$$

You can verify that these states are orthogonal to $\frac{1}{\sqrt{2}}|1\,1\rangle - \frac{1}{\sqrt{2}}|1\ {-1}\rangle$.

The normalized state is $\boxed{|\psi\rangle = \frac{1}{2}\left[|1\,1\rangle \pm \sqrt{2}\,|1\,0\rangle + |1\,{-1}\rangle\right]}$

Then

$$L_x|\psi\rangle = \frac{\hbar}{4}\left[\sqrt{2}\,|1\ 0\rangle \pm 2|1\ 1\rangle \pm 2|1\ {-1}\rangle + \sqrt{2}\,|1\,0\rangle\right]$$

$$\boxed{L_x|\psi\rangle = \pm\hbar\,|\psi\rangle}$$

3 (a) $\frac{p^2}{2m}\psi = E\psi$ radial equation, $l=0$ $V=0$

page 4

$$-\frac{\hbar}{2m}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\right)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}\right] = E\psi \qquad (1)$$

Try $\psi = \frac{e^{ikr}}{r}$ $\qquad \frac{d\psi}{dr} = ik\psi - \frac{1}{r}\psi \qquad (2)$

$$\frac{d^2\psi}{dr} = ik\frac{d\psi}{dr} - \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\psi$$

$$\frac{d^2\psi}{dr^2} = -k^2\psi - \frac{ik}{r}\psi - \frac{ik}{r}\psi + \frac{1}{r^2}\psi + \frac{1}{r^2}\psi$$

$$\frac{d^2\psi}{dr^2} = -k^2\psi - \frac{2ik}{r}\psi + \frac{2}{r^2}\psi \qquad (3)$$

Putting equations (2) and (3) into (1)

$$-\frac{\hbar^2}{2m}\left[-k^2\psi - \frac{2ik\psi}{r} + \frac{2\psi}{r^2} + \frac{2ik\psi}{r} - \frac{2\psi}{r^2}\right] = E\psi$$

$$\frac{k^2\hbar^2}{2m}\psi = E\psi \qquad \boxed{k = \pm\frac{\sqrt{2Em}}{\hbar}}$$

page 5

(b) $\psi = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}$

$r\psi = A\cos kr + iA\sin kr + B\cos kr - iB\sin kr$

$\psi(r) = \left(\frac{A+B}{r}\right)\cos kr + i\left(\frac{A-B}{r}\right)\sin kr$

$\psi(r) = \frac{C}{r}\cos kr + \frac{D}{r}\sin kr \quad \begin{pmatrix} C = A+B \\ D = i(A-B) \end{pmatrix}$

In order to keep $\psi(r)$ finite at $r=0$, $C=0$

$\psi(r) = D\frac{\sin kr}{r}$

Want $\psi(a)=0 \Rightarrow ka = n\pi \quad (n=1,2,3\ldots)$

$k^2 = \frac{n^2\pi^2}{a^2} \qquad \frac{2Em}{\hbar^2} = \frac{n^2\pi^2}{a^2}$

$$\boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

4 (a) $L_x = \frac{1}{2}(L_+ + L_-)$

$$2\langle L_x\rangle = \langle \ell m|L_x|\ell m\rangle = \langle \ell m|L_+|\ell m\rangle + \langle \ell m|L_-|\ell m\rangle$$

$$\langle L_x\rangle = C_+\langle \ell m|\ell\, m+1\rangle + C_-\langle \ell m|\ell\, m-1\rangle$$

By the orthogonality of the states

$$\boxed{\langle L_x\rangle = 0}$$

(b) $L_x^2 = \frac{1}{4}(L_+ + L_-)(L_+ + L_-)$

$$\langle L_x^2\rangle = \frac{1}{4}\langle \ell m|L_-L_+|\ell m\rangle + \frac{1}{4}\langle \ell m|L_-L_+|\ell m\rangle$$

(because $\langle \ell m|L_\pm L_\pm|\ell m\rangle = 0$

Then

$$\langle L_x^2\rangle = \frac{\hbar}{4}\sqrt{\ell(\ell+1)-m(m+1)}\,\langle \ell m|L_-|\ell\, m+1\rangle +$$
$$\frac{\hbar}{4}\sqrt{\ell(\ell+1)-m(m-1)}\,\langle \ell m|L_+|\ell\, m-1\rangle$$

$$\langle L_x^2\rangle = \frac{\hbar^2}{4}\sqrt{\ell(\ell+1)-m(m+1)}\,\sqrt{\ell(\ell+1)-(m+1)(m+1-1)}\,\langle \ell m|\ell m\rangle$$
$$+\frac{\hbar^2}{4}\sqrt{\ell(\ell+1)-m(m-1)}\,\sqrt{\ell(\ell+1)-(m-1)(m-1+1)}\,\langle \ell m|\ell m\rangle$$

Since $\langle e m | e m \rangle = 1$

page 7

$$\langle L_x^2 \rangle = \frac{\hbar^2}{4}\left[\ell(\ell+1) - m(m+1) + \ell(\ell+1) - m(m-1) \right]$$

$$\langle L_x^2 \rangle = \frac{\hbar^2}{4}\left[2\ell(\ell+1) - 2m^2 \right]$$

$$\boxed{\langle L_x^2 \rangle = \frac{1}{2}\hbar^2\left[\ell(\ell+1) - m^2 \right]}$$