1. (a) If the particles are distinguishable, there is no symmetrization requirement, so we have $\psi_d(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)$.

   (b) If the particles are indistinguishable bosons, the wavefunction must be symmetric under exchange, so we have
$$\psi_b(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) + \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right]$$

   (c) If the particles are indistinguishable fermions, the wavefunction must be antisymmetric under exchange, so we have
$$\psi_f(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) - \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right]$$

   (d) If the states are the same, the wavefunction is just $\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)$. Such a state is allowed.

   (e) For fermions, the two terms in the antisymmetrized wavefunction cancel if $a = b$. Two indistinguishable fermions thus cannot be in the same state.

2. (a) For bosons, the total wavefunction must be symmetric under exchange, so an antisymmetric spatial part requires an antisymmetric spin part. For fermions, the total wavefunction must be antisymmetric under exchange, so a symmetric spin part requires an antisymmetric spatial part.

   (b) The states $|1, 1\rangle = |\uparrow\uparrow\rangle$ and $|-1, -1\rangle = |\downarrow\downarrow\rangle$ are the simplest; the two electrons are in the same state, so these are clearly symmetric. The state
$$|1, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$
   is also symmetric under exchange; swapping the electrons swaps the two terms inside the parentheses, but since they're added together this gives us back the original state. The state
$$|0, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$
   is antisymmetric under exchange; swapping the electrons again swaps the two terms inside the parentheses, but since they're subtracted this adds an overall minus sign. All four thus have a well-defined symmetry under exchange.

   (c) For the three symmetric states with $s = 1$, the spatial wavefunction must be antisymmetric under exchange. For the antisymmetric $s = 0$ state, the spatial wavefunction must be symmetric under exchange.

   (d) We know that
$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|1, 0\rangle + |0, 0\rangle\right)$$
   The first term needs to come with an antisymmetric spatial part
$$\phi_-(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}\left(\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) - \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\right)$$
   while the second needs to come with a symmetric spatial part
$$\phi_+(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}\left(\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) + \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\right)$$
   so the total wavefunction is given by
$$\psi_{10} = \frac{1}{\sqrt{2}}\left(\phi_- |1, 0\rangle + \phi_+ |0, 0\rangle\right)$$

(e) To answer this, we can expand the expression above and look for the spin of the electron in spatial state ϕ_a in each of the terms in the linear combination. We have

$$\begin{aligned}
\psi_{10} &= \frac{1}{\sqrt{2}}\left(\phi_- \left|1,0\right\rangle + \phi_+ \left|0,0\right\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left[\left(\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) - \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\right)\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) + \left(\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) + \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\right)\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)\right] \\
&= \frac{1}{2\sqrt{2}}\left[(1+1)\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2)\left|\uparrow\downarrow\right\rangle + (1-1)\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2)\left|\downarrow\uparrow\right\rangle - (1-1)\phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\left|\uparrow\downarrow\right\rangle - (1+1)\phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\left|\downarrow\uparrow\right\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2)\left|\uparrow\downarrow\right\rangle - \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\left|\downarrow\uparrow\right\rangle\right]
\end{aligned}$$

We see that no matter whether we measured the first or second particle, if we measured it in ϕ_a, then it is spin-up with 100% probability.

(f) We can't say the first electron is spin-up because we know of way of establishing that one of those particles is the first electron; because of (anti)symmetrization, any electron we select could equally well be the second electron. However, we can say that the spin-up electron is always in ϕ_a, and the spin-down electron is always in ϕ_b. While we haven't written a state proportional to $\left|\uparrow\downarrow\right\rangle$ (and never can, due to exchange), we have written a state where which electron is spin-up is correlated with another quality: the position, which is something we can measure and determine.

3. (a) If the particles are distinguishable, they can each independently be in any of the three states a, b, c. There are thus $3 \times 3 = 9$ possible two particle states.

(b) If the particles are indistinguishable bosons, if we pick different states for our two bosons, it does not matter which we assign to the first and which to the second; symmetrization will mean these gives the same state in the end. There are three possibilities with the same state for both particles and $6/2 = 3$ possibilities with different states for the two particles, giving six possible two particle states.

(c) If the particles are indistinguishable fermions, it once again does not matter which particle is assigned which state if we pick different states for the two particles. In addition, per 1(e), we cannot have both particles in the same state. There are thus only three possible two particle states for indistinguishable fermions.