Physics 115B, Mastery Questions for Section 7 Spring 22

Today we are going to practice working with identical particles.

1. Consider a system of two non-interacting particles, with the first particle in the state $\psi_a(\mathbf{r}_1)$ and the second particle in the state $\psi_b(\mathbf{r}_2)$.

   (a) Write down the wavefunction for the system, $\psi(\mathbf{r}_1, \mathbf{r}_2)$, in terms of the wavefunctions of the constituent particles, assuming the particles are distinguishable.

   (b) What is $\psi(\mathbf{r}_1, \mathbf{r}_2)$ if the particles are indistinguishable bosons?

   (c) What is $\psi(\mathbf{r}_1, \mathbf{r}_2)$ if the particles are indistinguishable fermions?

   (d) Now, consider the case that the two particles are in the same state, $a = b$. What is $\psi(\mathbf{r}_1, \mathbf{r}_2)$ if the particles are indistinguishable bosons? What conclusions can we draw?

   (e) Still with $a = b$, what is $\psi(\mathbf{r}_1, \mathbf{r}_2)$ if the particles are indistinguishable fermions? What conclusions can we draw?

2. Now, consider the complete state for a system, $\phi(\mathbf{r})\chi(\mathbf{s})$, which includes both the spatial wavefunction and the spinor state.

   (a) Recall the symmetrization requirement, which gives the solutions to the Schrödinger equation under exchange

   $$P_{12}\psi = \pm\psi \tag{1}$$

   where the plus sign corresponds to bosons and the minus sign corresponds to fermions. If we now consider the total wavefunction $\psi = \phi(\mathbf{r})\chi(\mathbf{s})$:

      i. If $\phi(\mathbf{r})$ is antisymmetric under exchange, should $\chi(\mathbf{s})$ be symmetric or antisymmetric if the particle is a boson?

      ii. If $\chi(\mathbf{s})$ is symmetric under exchange, should $\phi(\mathbf{r})$ be symmetric or antisymmetric if the particle is a fermion?

   (b) Now specialize to the case of two electrons. Of the four spin states in the coupled basis, which are symmetric under the exchange of particles? Which are antisymmetric? Which don't have a well-defined symmetry under exchange?

(c) For each of these four spin states, what must be true of the spatial wavefunction $\phi(\mathbf{r}_1, \mathbf{r}_2)$ under exchange?

(d) Suppose the spatial wavefunctions of each electron are $\phi_a(\mathbf{r})$ and $\phi_b(\mathbf{r})$ (i.e., the total spatial wavefunction could be $\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2), \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)$, or a linear combination thereof). Suppose as well that we want the spin state of our electrons to be $|\uparrow\downarrow\rangle$ (which is in the uncoupled basis). Since $|\uparrow\downarrow\rangle$ is neither symmetric nor antisymmetric under exchange, we can't immediately write down a total wavefunction $\psi = \phi(\mathbf{r}) |\uparrow\downarrow\rangle$. By writing $|\uparrow\downarrow\rangle$ as a linear combination of the coupled states $|10\rangle$ and $|00\rangle$, write down a possible total wavefunction ψ which has the appropriate properties under exchange.

(e) Using the wavefunction ψ from the previous part, suppose we measure the position of one of the electrons (but don't know which one) and fight that its spatial wavefunction is ϕ_a. What are the possible outcomes and their probabilities if we measure the spin of this electron?

(f) Can we say the first electron has spin up? What can we say about the spin-up electron?

3. Consider a system of two non-interacting particles, and three one-particle states $\psi_a(\mathbf{r}_1)$, $\psi_b(\mathbf{r}_2)$, $\psi_c(\mathbf{r}_3)$. Assuming these states are orthonormal, how many different two-particle states can be constructed if the particles are:

(a) Distinguishable?

(b) Indistinguishable bosons?

(c) Indistinguishable fermions?