1. (a) The z-component of the total spin is given by adding the z-components of the constituents,

$$m = m_1 + m_2 = 1/2.$$

This is not a state of definite total spin; S^2 does not commute with S_{1z} and S_{2z}. To get a state of definite total spin, we would need a particular linear combination (which can be read off a Clebsch-Gordan table) of $m_1 = 1, m_2 = -1/2$ and $m_1 = 0, m_2 = 1/2$.

(b) We need to add together three angular momenta: the graviton spin, the electron spin, and the orbital angular momentum. The graviton and photon spins together can have

$$s = 2 + 1/2 = 5/2 \quad \text{or} \quad s = 2 - 1/2 = 3/2,$$

For $s = 5/2$, the total angular momentum is $j = 5/2 + 0 = 5/2$. For $s = 3/2$, the total angular momentum is $j = 3/2 + 0 = 3/2$. Sensibly, when there is no orbital angular momentum, the total angular momentum is just given by the total spin.

$j = 5/2$ has $m = \pm 5/2, \pm 3/2, \pm 1/2$ as possibilities; $j = 3/2$ has $m = \pm 3/2, \pm 1/2$ as possibilities. In total, the possible z-components or the total angular momentum are thus $m = \pm 5/2$ with multiplicity one and $m = \pm 3/2, \pm 1/2$ with multiplicity two each. Note that we have ten states, the same as the 5 (for the graviton) times 2 (for the electron) there are in the uncoupled basis. We could also have figured the possible z-components from the uncoupled basis and the m_1 and m_2 values.

(c) We now have a non-trivial addition of the third angular momentum. It is still the case that the total spin is $s = 5/2$ or $s = 3/2$. If it is 5/2, then j is between

$$j_{\max} = 5/2 + 1 = 7/2 \quad \text{and} \quad j_{\min} = 5/2 - 1 = 3/2.$$

If $s = 3/2$, then j is between

$$j_{\max} = 3/2 + 1 = 5/2 \quad \text{and} \quad j_{\min} = 3/2 - 1 = 1/2.$$

The complete list of possibilities for j is thus $^1/_2, ^3/_2, ^5/_2, ^7/_2$, with two different ways to make each of $j = 3/2, 5/2$. The possible z components of angular momentum are thus $m = \pm 1/2$ with multiplicity 6 (each of our four possible j can have $m = 1/2$, and two of them occur twice); $m = \pm 3/2$ with multiplicity 5 (each of the possible $j \geq 3/2$); $m = \pm 5/2$ with multiplicity 3; and $m = \pm 7/2$ with multiplicity 1. We can check that the total number of states is $2 \times (6 + 5 + 3 + 1) = 30$, the same as the $5 \times 2 \times 3$ in the uncoupled representation. We could again have figured the possible z-components from the uncoupled basis and the m_1, m_2, and m_3 values.

2. (a) We have

$$\mathbf{E}' = -\boldsymbol{\nabla}\phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \boldsymbol{\nabla}\Lambda}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}$$

since the partial derivative with respect to time commutes with the partial derivatives with respect to space in the gradient. We also have

$$\mathbf{B}' = \boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\Lambda = \boldsymbol{\nabla} \times \mathbf{A}$$

since the curl of a gradient vanishes.

(b) As long as the electric and magnetic fields are the same, all physical effects will be the same. We thus cannot determine what Λ is; the potentials $\phi, \mathbf{A}$ are only defined up to the gauge transformation, and different choices of Λ are physically irrelevant and will give the same answer.

3. Classically the relative orbital angular momentum is zero because the outgoing particles are traveling back-to-back. The total angular momentum is going to be the sum of the spins of the outgoing particles and the relative orbital angular mometum, ie, $J = S_1 \otimes S_2 \otimes L = 0 \otimes 1/2 \otimes L = 1/2 \otimes L = (L + 1/2) \oplus (L - 1/2)$ (if L=0 the second term here does not exist). Conservation of angular momentum implies that J equal the spin of the decaying particle, i.e., $J = 3/2$. Thus $L = 1$ or $L = 2$ are possible and $L = 0$ is not allowed by conservation of angular momentum.