1. (a) There was no question here, just preamble.

(b) On the left-hand side, we have

$$S_- |1\,1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\,|1\,1-1\rangle = \sqrt{2}\hbar\,|1\,0\rangle\,.$$

On the right-hand side, we have

$$\begin{aligned}(S_{1-} + S_{2-})\,|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle &= S_{1-}\,|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle + S_{2-}\,|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle \\ &= \hbar\sqrt{\frac{1}{2}\frac{3}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)}\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle + \hbar\sqrt{\frac{1}{2}\frac{3}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)}\,|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle \\ &= \hbar\left(|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle + |{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\right).\end{aligned}$$

Dividing both sides by $\sqrt{2}\hbar$ gives

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle + \frac{1}{\sqrt{2}}\,|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,.$$

For the first uncoupled state on the right, we have $m_1 + m_2 = -1/2 + 1/2 = 0 = m$; for the second, we have $m_1 + m_2 = 1/2 - 1/2 = 0 = m$. All is as it should be.

(c) On the left-hand side, we have

$$S_- |1\,0\rangle = \hbar\sqrt{1(1+1) - 0(0-1)}\,|1\,0-1\rangle = \sqrt{2}\hbar\,|1\ {-1}\rangle$$

On the right-hand side, we have a sum applied to a sum, so we get four terms. However, we can't lower a state that has $m_{1/2} = -1/2$. We thus have

$$\begin{aligned}(S_{1-} + S_{2-})\frac{1}{\sqrt{2}}\left(|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle + |{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\right) &= \frac{1}{\sqrt{2}}\left(S_{2-}\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle + S_{1-}\,|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\right) \\ &= \frac{1}{\sqrt{2}}\hbar\left(|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle + |{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\right) \\ &= \sqrt{2}\hbar\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,.\end{aligned}$$

Dividing both sides by $\sqrt{2}\hbar$ gives

$$|1\ {-1}\rangle = |{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,.$$

We indeed have $m = m_1 + m_2$. Notice as well that this is the mirror of $|1\,1\rangle$, with all z-components flipped.

(d) Since we have $m_{1/2} = \pm 1/2$, we can have $m = 1, 0, -1$. There is only one uncoupled state for each of $m = \pm 1$, and we already have coupled states equal to these; we can't write another and have it be orthogonal. For $m = 0$, there are two uncoupled states, and we have only one coupled state. We thus expect our fourth coupled state to be a linear combination of $|{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle$ and $|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle$.

(e) As suggested, our fourth state should have $s < 1$, so it could be $1/2$ or 0. To have $m = 0$, the only one which works is $s = 0$.

(f) Our existing state with $m = 0$ has the two coefficients of its linear combination positive and of equal magnitude. We can write an orthogonal state by making the coefficients have opposite sign; our fourth coupled state is then

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|{}^1\!/_2\,{}^1\!/_2\rangle\,|{}^1\!/_2\ {-}{}^1\!/_2\rangle - |{}^1\!/_2\ {-}{}^1\!/_2\rangle\,|{}^1\!/_2\,{}^1\!/_2\rangle\right)$$

If you like, you can check that this linear combination really does have $s = 0$ by applying S^2 and seeing that you get 0.

2. (a) In the $1 \times 1/2$ table, we can find the column headed by 3/2 1/2. There are two entries, the first for 1 -1/2 (meaning $|1\,1\rangle\,|{}^1\!/\!{}_2\ {-}{}^1\!/\!{}_2\rangle$) and the second for 0 1/2 (meaning $|1\,0\rangle\,|{}^1\!/\!{}_2\,{}^1\!/\!{}_2\rangle$). We have

$$|{}^3\!/\!{}_2\,{}^1\!/\!{}_2\rangle = \sqrt{\frac{1}{3}}\,|1\,1\rangle\,|{}^1\!/\!{}_2\ {-}{}^1\!/\!{}_2\rangle + \sqrt{\frac{2}{3}}\,|1\,0\rangle\,|{}^1\!/\!{}_2\,{}^1\!/\!{}_2\rangle$$

In the $1 \times 1/2$ table, we can find the row headed by 0 1/2. There are two entries, the first for $|{}^3\!/\!{}_2\,{}^1\!/\!{}_2\rangle$ and the second for $|{}^1\!/\!{}_2\,{}^1\!/\!{}_2\rangle$. We have

$$|1\,0\rangle\,|{}^1\!/\!{}_2\,{}^1\!/\!{}_2\rangle = \sqrt{\frac{2}{3}}\,|{}^3\!/\!{}_2\,{}^1\!/\!{}_2\rangle - \sqrt{\frac{1}{3}}\,|{}^1\!/\!{}_2\,{}^1\!/\!{}_2\rangle$$

(b) In the 1×1 table, we can find the column headed by 2 0. There are three entries, giving

$$|2\,0\rangle = \sqrt{\frac{1}{6}}\,|1\,1\rangle\,|1\ {-1}\rangle + \sqrt{\frac{2}{3}}\,|1\,0\rangle\,|1\,0\rangle + \sqrt{\frac{1}{6}}\,|1\ {-1}\rangle\,|1\,1\rangle$$

In the 1×1 table, we can find the row headed by 1 -1. There are three entries, giving

$$|1\,1\rangle\,|1\ {-1}\rangle = \sqrt{\frac{1}{6}}\,|2\,0\rangle + \sqrt{\frac{1}{2}}\,|1\,0\rangle + \sqrt{\frac{1}{3}}\,|0\,0\rangle\,.$$