1. (a) There was no question here, just preamble.

(b) On the left-hand side, we have

$$S_- |11\rangle = \hbar \sqrt{1(1 + 1) - 1(1 - 1)} |11 - 1\rangle = \sqrt{2}\hbar |10\rangle.$$  

On the right-hand side, we have

$$(S_1 - + S_2 -) |1/2 \pm 1/2 \rangle |1/2 \pm 1/2 \rangle = S_1 - |1/2 \pm 1/2 \rangle |1/2 \pm 1/2 \rangle + S_2 - |1/2 \pm 1/2 \rangle |1/2 \pm 1/2 \rangle$$

$$= \hbar \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle + \hbar \left( \frac{1}{2} - \frac{1}{2} \right) |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle$$

$$= \hbar |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle + |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle.$$  

Dividing both sides by $\sqrt{2}\hbar$ gives

$$|10\rangle = \frac{1}{\sqrt{2}} |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle + \frac{1}{\sqrt{2}} |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle.$$  

For the first uncoupled state on the right, we have $m_1 + m_2 = -1/2 + 1/2 = 0 = m$; for the second, we have $m_1 + m_2 = 1/2 - 1/2 = 0 = m$. All is as it should be.

(c) On the left-hand side, we have

$$S_- |10\rangle = \hbar \sqrt{1(1 + 1) - 0(0 - 1)} |10 - 1\rangle = \sqrt{2}\hbar |1 - 1\rangle.$$  

On the right-hand side, we have a sum applied to a sum, so we get four terms. However, we can’t lower a state that has $m_{1/2} = -1/2$. We thus have

$$(S_1 - + S_2 -) \frac{1}{\sqrt{2}} (|1/2 - 1/2 \rangle |1/2 - 1/2 \rangle + |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle) = \frac{1}{\sqrt{2}} (S_2 - |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle + S_1 - |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle)$$

$$= \frac{1}{\sqrt{2}} \hbar (|1/2 - 1/2 \rangle |1/2 - 1/2 \rangle + |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle)$$

$$= \sqrt{2}\hbar |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle.$$  

Dividing both sides by $\sqrt{2}\hbar$ gives

$$|1 - 1\rangle = |1/2 - 1/2 \rangle |1/2 - 1/2 \rangle.$$  

We indeed have $m = m_1 + m_2$. Notice as well that this is the mirror of $|11\rangle$, with all $z$-components flipped.

(d) Since we have $m_{1/2} = \pm 1/2$, we can have $m = 1, 0, -1$. There is only one uncoupled state for each of $m = \pm 1$, and we already have coupled states equal to these; we can’t write another and have it be orthogonal. For $m = 0$, there are two uncoupled states, and we have only one coupled state. We thus expect our fourth coupled state to be a linear combination of $|1/2 - 1/2 \rangle |1/2 - 1/2 \rangle$ and $|1/2 - 1/2 \rangle |1/2 - 1/2 \rangle$.

(e) As suggested, our fourth state should have $s < 1$, so it could be $1/2$ or $0$. To have $m = 0$, the only one which works is $s = 0$.

(f) Our existing state with $m = 0$ has the two coefficients of its linear combination positive and of equal magnitude. We can write an orthogonal state by making the coefficients have opposite sign; our fourth coupled state is then

$$|00\rangle = \frac{1}{\sqrt{2}} (|1/2 1/2 \rangle |1/2 - 1/2 \rangle - |1/2 - 1/2 \rangle |1/2 1/2 \rangle)$$

If you like, you can check that this linear combination really does have $s = 0$ by applying $S^2$ and seeing that you get $0$.  

2. (a) In the $1 \times 1/2$ table, we can find the column headed by $3/2, 1/2$. There are two entries, the first for $1 -1/2$ (meaning $|1\,1\rangle, |1/2\rangle$) and the second for $0\,1/2$ (meaning $|1\,0\rangle, |1/2\rangle$). We have

$$|3/2, 1/2\rangle = \sqrt{1 \over 3} |1\,1\rangle|1/2\rangle - \sqrt{2 \over 3} |1\,0\rangle|1/2\rangle.$$ 

In the $1 \times 1/2$ table, we can find the row headed by $0\,1/2$. There are two entries, the first for $|3/2, 1/2\rangle$ and the second for $|1/2, 1/2\rangle$. We have

$$|1\,0\rangle|1/2\rangle = \sqrt{2 \over 3} |3/2, 1/2\rangle - \sqrt{1 \over 3} |1/2, 1/2\rangle.$$ 

(b) In the $1 \times 1$ table, we can find the column headed by $2, 0$. There are three entries, giving

$$|2\,0\rangle = \sqrt{1 \over 6} |1\,1\rangle|1\rangle - |1\rangle + \sqrt{2 \over 3} |1\,0\rangle|0\rangle + \sqrt{1 \over 6} |1\rangle - |1\rangle.$$ 

In the $1 \times 1$ table, we can find the row headed by $1, -1$. There are three entries, giving

$$|1\,1\rangle|1\rangle - |1\rangle = \sqrt{1 \over 6} |2\,0\rangle + \sqrt{1 \over 2} |1\,0\rangle + \sqrt{1 \over 3} |0\rangle.$$