Physics 115B Mastery Questions for Section 5 Spring 22

1. Suppose we have a system comprising two spin-1/2 particles. In lecture, we discussed finding a relation between the uncoupled representation, which is written in terms of the spins and z-components of the individual particles $|s_1\, m_1\rangle\, |s_2\, m_2\rangle$, and the coupled representation, which is written in terms of the total spin and z-component thereof $|s\, m\rangle$. This problem will walk you through deriving these relations.

   (a) Recall that in lecture we found the relation

   $$|1\,1\rangle = |{}^1\!/\!_2\, {}^1\!/\!_2\rangle\, |{}^1\!/\!_2\, {}^1\!/\!_2\rangle\, .$$

   On physics grounds, we can understand why this is so: the z-component of the total is just equal to the sum of the z-components of the individual particles, $m = m_1 + m_2$; for $m = 1$, we must have $s \geq 1$; and we should have $s \leq s_1 + s_2$, since the total spin can't be more than the sum of the spins of the individual particles (although it could be less, since they could partially or totally cancel).

   (b) We can find another state in the coupled representation by acting with the lowering operator $S_- = S_{1-} + S_{2-}$. Recall that the lowering operator satisfies

   $$S_-\, |s\, m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\, |s\, m-1\rangle\, .$$

   so acting with it on $|1\,1\rangle$ will give us something proportional to $|1\,0\rangle$. On the right-hand side, acting with $S_{1-} + S_{2-}$ will give us a linear combination of uncoupled states. Check that the relation you've found satisfies $m = m_1 + m_2$.

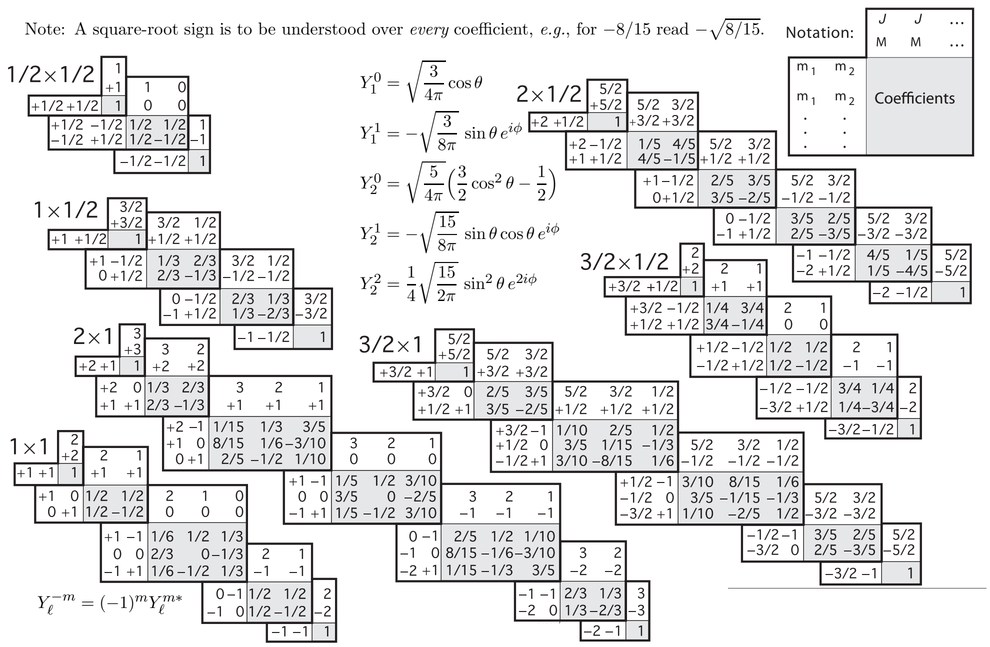
   (c) Act with the lowering operator again to find a third state in the coupled representation, and check that it satisfies $m = m_1 + m_2$.

   (d) There are four independent states in the uncoupled representation, so there are also four independent states in the coupled representation. We now have three states in the coupled representation expressed in terms of the uncoupled representation. If we try to apply the lowering operator a third time, though, we won't get a fourth state; our third state already has the smallest m possible. To get the fourth state, we need a new method. To start, which of the uncoupled states could the fourth state in the coupled representation be composed of? Remember that $m = m_1 + m_2$ and that this fourth state should be orthogonal to the three we've found so far.

   (e) All three states we've found so far have $s = 1$. We expect there should be at least one state with $s < 1$; the spins of our two particles don't need to be in the same direction, and if they're not they will at least partially cancel. Given the m value of the fourth state you found above, what must the value of s be?

   (f) Use orthogonality to solve for the fourth state.

What you've derived are a specific case of Clebsch-Gordan coefficients; these are the coefficients for the linear combination of uncoupled states which will give you a particular coupled state (or vice versa). Finding these coefficients for higher spin is analogous to the procedure you followed above: start with the largest m-value, apply the lowering operator as much as you can, and use orthogonality as necessary. Thankfully this has been done, and the results can be presented in a giant table:



If you want to find a particular coupled state in terms of uncoupled states, you find the column headed by the coupled state you want and read down the column. Each entry is the coefficient of the uncoupled state at the beginning of that row. Be careful: each of the coefficients is missing a square root to save space in the tables. You should find that the table for $1/2 \times 1/2$ agrees with your results. Conversely, if you want an uncoupled state in terms of coupled states, you find the row for the uncoupled state you want and read across.

2. Now let's get some practice reading the table.

   (a) For a spin-1 and spin-$1/2$ particle, write $|3/2\, 1/2\rangle$ in terms of uncoupled states and $|1\,0\rangle\,|1/2\, 1/2\rangle$ in terms of coupled states.

   (b) For a pair of spin-1 particles, write $|2\,0\rangle$ in terms of uncoupled states and $|1\,1\rangle\,|1\,-1\rangle$ in terms of coupled states.

Note: Here we have talked about combining spins of two different particles, but remember that spin is just one type of angular momentum. The exact same discussion applies to combining orbital angular momenta of two different particles. It also applies to combining the spin and the orbital angular momentum of the same particle to yield the total angular momentum of that particle.