1. (a) The spin operators are given by
$$S_i = \frac{\hbar}{2}\sigma_i$$
You can check this quickly by looking at the z-component, since it's diagonal; with the factor of $\hbar/2$, the two eigenvalues are $\pm\hbar/2$, exactly as they should be for spin-1/2.

   (b) The raising and lowering operators are given by $S_\pm = S_x \pm iS_y$. We thus have
$$S_+ = \frac{\hbar}{2}(\sigma_x + i\sigma_y) = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad S_- = \frac{\hbar}{2}(\sigma_x - i\sigma_y) = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$
Neither of these are Hermitian; they do not correspond to observables.

   (c) We have
$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

2. (a) A normalized state satisfies
$$1 = \langle\chi|\chi\rangle = A\begin{pmatrix}-i & -2\end{pmatrix}A\begin{pmatrix}i\\ -2\end{pmatrix} = A^2\left(-i^2 + (-2)^2\right) = 5A^2 \implies A = 1/\sqrt{5}$$

   (b) The expectation values we need are
$$\begin{aligned}
\langle S_x\rangle = \langle\chi| S_x |\chi\rangle &= \begin{pmatrix}-i/\sqrt{5} & -2/\sqrt{5}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}i/\sqrt{5}\\ -2/\sqrt{5}\end{pmatrix} = \frac{\hbar}{2}(2i/5 - 2i/5) = 0\\
\langle S_x^2\rangle = \langle\chi| S_xS_x |\chi\rangle &= \begin{pmatrix}-i/\sqrt{5} & -2/\sqrt{5}\end{pmatrix}\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}i/\sqrt{5}\\ -2/\sqrt{5}\end{pmatrix} = \frac{\hbar^2}{4}\\
\langle S_y\rangle = \langle\chi| S_y |\chi\rangle &= \begin{pmatrix}-i/\sqrt{5} & -2/\sqrt{5}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}i/\sqrt{5}\\ -2/\sqrt{5}\end{pmatrix} = \frac{\hbar}{2}(2/5 + 2/5) = \frac{2\hbar}{5}\\
\langle S_y^2\rangle = \langle\chi| S_yS_y |\chi\rangle &= \begin{pmatrix}-i/\sqrt{5} & -2/\sqrt{5}\end{pmatrix}\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}i/\sqrt{5}\\ -2/\sqrt{5}\end{pmatrix} = \frac{\hbar^2}{4}\\
\langle[S_x, S_y]\rangle = \langle i\hbar S_z\rangle &= \begin{pmatrix}-i/\sqrt{5} & -2/\sqrt{5}\end{pmatrix}\frac{i\hbar^2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}i/\sqrt{5}\\ -2/\sqrt{5}\end{pmatrix} = \frac{i\hbar^2}{2}(1/5 - 4/5) = -\frac{3i\hbar^2}{10}
\end{aligned}$$
We thus have
$$\begin{aligned}
\Delta S_x &= \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\hbar^2/4 - 0} = \hbar/2\\
\Delta S_y &= \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\hbar^2/4 - 4\hbar^2/25} = 3\hbar/10
\end{aligned}$$
and
$$\Delta S_x \Delta S_y = \frac{3\hbar^2}{20} = \frac{3\hbar^2}{20} = \frac{1}{2}|\langle[S_x, S_y]\rangle|$$
so this satisfies the uncertainty principle and is a minimum uncertainty state.

   (c) The possible outcomes are the eigenvalues of S_z, $\pm\hbar/2$. The probability of each is given by the magnitude-squared of the coefficient of that eigenstate. We could thus get $\hbar/2$ with probability 1/5 or $-\hbar/2$ with probability 4/5.

   (d) After a measurement of S_z as $\hbar/2$, the system has collapsed to the state χ_+, the eigenstate of S_z with eigenvalue $\hbar/2$.

   (e) Before the measurement, the system was in the state χ given above (*not* χ_+).

3. (a) The coefficient of χ_+ should be positive. $\cos\alpha$ takes on all values between 0 and 1 as α goes from 0 to $\pi/2$, so the range of α is $[0, \pi/2]$. $e^{i\delta}$ is periodic with period 2π, so the range of δ is $[0, 2\pi)$.

   (b) We can determine the value of α from the first component; it is $\alpha = \pi/4$ for the x and y eigenstates; $\alpha = 0$ for $\chi_{z,+}$; and $\alpha = \pi/2$ for $\chi_{z,-}$. The value of δ is determined by the phase of the second component; it is $\delta = 0$ for $\chi_{x,+}$; $\delta = \pi$ for $\chi_{x,-}$; $\delta = \pi/2$ for $\chi_{y,+}$; $\delta = 3\pi/2$ for $\chi_{y,-}$; and indeterminate for the two z eigenstates.

   (c) From above, we have $\alpha = 0, \pi/2$ for the two z eigenstates. We want these to correspond to the north and south poles of the sphere, which are the points $\theta = 0$ and $\theta = \pi$. We thus want to identify $\alpha = \theta/2$. $\phi = 0$ is along the positive x axis, and ϕ increases counterclockwise, so $\phi = \pi/2$ is along the y axis. We see this matches exactly with the δ values computed above, so we identify $\delta = \phi$.