Physics 115B, Mastery Questions for Section 4 Spring 22

In today’s section, we will develop our knowledge of spin-1/2 systems and consider how spin-1/2 particles behave in magnetic fields.

1. Recall the Pauli spin matrices
\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (1)

(a) What are the spin operators \( S_x, S_y, S_z \) in terms of Pauli matrices?
(b) Construct the raising and lowering operators \( S_+ \) and \( S_- \) in terms of Pauli matrices. Are these Hermitian? Why or why not?
(c) Using your results so far, construct the matrix for the \( S^2 \) operator.

2. Recall that we can represent a particle in the spin up and spin down state (in the \( S_z \) basis) as, respectively
\[ \chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (2)

Consider a particle in the state \( |\chi\rangle = A(i\chi^+ - 2\chi^-) \).

(a) Normalize this spin state by finding an appropriate value of \( A \).
(b) Compute the standard deviations \( \Delta S_x \) and \( \Delta S_y \) and check that your results agree with the uncertainty principle.
(c) If a measurement is made on the system of \( S_z \), what are the possible outcomes and with what probabilities?
(d) A particular system, when measured, gives the result \( \frac{\hbar}{2} \) for \( S_z \). In what state is the system after the measurement?
(e) What can we say about the state of that particular system before the measurement?

3. Recall that we can write any spin-1/2 particle as
\[ \chi = c_+\chi^+ + c_-\chi^- \approx \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \] (3)
where if the state is normalized, we must have $|c_+|^2 + |c_-|^2 = 1$. Since we have the sum of two squares equaling 1, this suggests a natural parametrization in terms of trig functions,

$$\chi = \cos(\alpha) \chi_+ + \sin(\alpha) e^{i\delta} \chi_-$$

where we only have a complex phase on the second coefficient since it’s only the phase difference between the components, not the overall phases, which has physical meaning.

(a) What are the ranges of $\alpha$ and $\delta$? Remember that we’ve moved the phase entirely to $c_-$, so $c_+$ should be real and positive.

(b) Recall that the eigenstates of $S_x, S_y, S_z$ are given by

$$\begin{align*}
\chi_{x,+} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \chi_{x,-} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\chi_{y,+} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, & \chi_{y,-} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
\chi_{z,+} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \chi_{z,-} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}$$

What are the values of $\alpha$ and $\delta$ for each of these six states?

(c) Our representation has two angles $\alpha$ and $\delta$; we can relate these to the spherical coordinates $\theta$ and $\phi$. A given state $\chi$ is then identified with a point on the unit sphere. Since spin is a vector, we can hope that the states are identified with sensible points on the sphere; for example, $\chi_{z,+}$ should be identified with the north pole, $\chi_{x,-}$ with the leftmost point on the equator, and so on. What is the necessary relation between $\alpha, \delta$ and $\theta, \phi$? This sphere is then known as the Bloch sphere.