Physics 115B, Mastery Questions for Section 4 Spring 22

In today's section, we will develop our knowledge of spin-1/2 systems and consider how spin-1/2 particles behave in magnetic fields.

1. Recall the Pauli spin matrices

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

   (a) What are the spin operators S_x, S_y, S_z in terms of Pauli matrices?

   (b) Construct the raising and lowering operators S_+ and S_- in terms of Pauli matrices. Are these Hermitian? Why or why not?

   (c) Using your results so far, construct the matrix for the $\mathbf{S}^2$ operator.

2. Recall that we can represent a particle in the spin up and spin down state (in the S_z basis) as, respectively

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2}$$

   Consider a particle in the state $|\chi\rangle = A(i\chi_+ - 2\chi_-)$.

   (a) Normalize this spin state by finding an appropriate value of A.

   (b) Compute the standard deviations ΔS_x and ΔS_y and check that your results agree with the uncertainty principle.

   (c) If a measurement is made on the system of S_z, what are the possible outcomes and with what probabilities?

   (d) A particular system, when measured, gives the result $\frac{\hbar}{2}$ for S_z. In what state is the system *after* the measurement?

   (e) What can we say about the state of that particular system *before* the measurement?

3. Recall that we can write any spin-1/2 particle as

$$\chi = c_+\chi_+ + c_-\chi_- \doteq \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \tag{3}$$

where if the state is normalized, we must have $|c_+|^2 + |c_-|^2 = 1$. Since we have the sum of two squares equaling 1, this suggests a natural parametrization in terms of trig functions,

$$\chi = \cos(\alpha)\chi_+ + \sin(\alpha)e^{i\delta}\chi_- \tag{4}$$

where we only have a complex phase on the second coefficient since it's only the phase difference between the components, not the overall phases, which has physical meaning.

(a) What are the ranges of α and δ? Remember that we've moved the phase entirely to c_-, so c_+ should be real and positive.

(b) Recall that the eigenstates of S_x, S_y, S_z are given by

$$\chi_{x,+} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \chi_{x,-} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{5}$$

$$\chi_{y,+} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad \chi_{y,-} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \tag{6}$$

$$\chi_{z,+} \doteq \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi_{z,-} \doteq \begin{pmatrix}0\\1\end{pmatrix} \tag{7}$$

What are the values of α and δ for each of these six states?

(c) Our representation has two angles α and δ; we can relate these to the spherical coordinates θ and ϕ. A given state χ is then identified with a point on the unit sphere. Since spin is a vector, we can hope that the states are identified with sensible points on the sphere; for example, $\chi_{z,+}$ should be identified with the north pole, $\chi_{x,-}$ with the leftmost point on the equator, and so on. What is the necessary relation between α, δ and θ, ϕ? This sphere is then known as the *Bloch sphere*.