1. The three components of linear momentum $\hat{p}_x, \hat{p}_y, \hat{p}_z$ commute with each other, so all three operators can be simultaneously diagonalized and we can label a state with all three eigenvalues. The three components of angular momentum do not commute. This means that a state cannot, in general, be a simultaneous eigenstate of all three and there is no well-defined angular momentum vector eigenvalue with which to label a state.

2. The lowering operator is the Hermitian conjugate of the raising operator and given by $\hat{L}_- = \hat{L}_x - i\hat{L}_y$. We can invert these relations to give $\hat{L}_x = (\hat{L}_+ + \hat{L}_-)/2$ and $\hat{L}_y = (\hat{L}_+ - \hat{L}_-)/2i$. To find $\langle L_x \rangle$ and $\langle L_y \rangle$, we thus need $\langle L_+ \rangle$ and $\langle L_- \rangle$. Both of these are zero due to the orthogonality of the eigenstates:

$$\langle l, m| \hat{L}_+ |l, m\rangle \propto \langle l, m|l, m+1\rangle = 0$$
$$\langle l, m| \hat{L}_- |l, m\rangle \propto \langle l, m|l, m-1\rangle = 0$$

We thus have $\langle L_x \rangle = \langle L_y \rangle = 0$. Alternatively, note that the angular momentum eigenstates have no preferred direction in the $x - y$ plane. They thus can't favor positive or negative L_x or L_y, and so the expectation values must be zero.

3. (a) We have $\langle L^2 \rangle = \hbar^2 l(l+1)$ for both cases. $\langle L_z^2 \rangle = \hbar^2 0^2 = 0$ for $|l, 0\rangle$ and $\langle L_z^2 \rangle = \hbar^2 l^2$ for $|l, l\rangle$.

   (b) We have

$$\begin{aligned}\langle L_x^2 \rangle &= \langle l, 0| \frac{(L_+ + L_-)^2}{4} |l, 0\rangle \\ &= \langle l, 0| \frac{L_+^2 + L_+L_- + L_-L_+ + L_-^2}{4} |l, 0\rangle \\ &= \frac{\hbar^2}{4}\left(0 + \sqrt{l(l+1)}^2 + \sqrt{l(l+1)}^2 + 0\right) \\ &= \frac{\hbar^2 l(l+1)}{2}\end{aligned}$$

   for $|l, 0\rangle$ and

$$\begin{aligned}\langle L_x^2 \rangle &= \langle l, l| \frac{(L_+ + L_-)^2}{4} |l, l\rangle \\ &= \langle l, l| \frac{L_+^2 + L_+L_- + L_-L_+ + L_-^2}{4} |l, l\rangle \\ &= \frac{\hbar^2}{4}\left(0 + \sqrt{l(l+1) - (l-1)l}\sqrt{l(l+1) - l(l-1)} + 0 + 0\right) \\ &= \frac{\hbar^2 l}{2}\end{aligned}$$

   for $|l, l\rangle$. By symmetry, $\langle L_y^2 \rangle$ is the same.

   (c) We $\Delta L_x = \Delta L_y = \hbar\sqrt{l(l+1)/2}$ for $|l, 0\rangle$ and $\Delta L_x = \Delta L_y = \hbar\sqrt{l/2}$ for $|l, l\rangle$. For the first, we see that $\Delta L_x, \Delta L_y$ are half the size of L^2; the angular momentum could be anywhere in the $x - y$ plane, with a magnitude corresponding to L^2. For the second, the uncertainty is not zero, but is parametrically smaller than L_z^2 (it scales as l rather than l^2). It's still symmetric between x, y, but now almost all of the angular momentum is in the z direction; as l becomes large, a vanishingly small proportion of the angular momentum is in the $x - y$ plane.

   (d) Recall that $[L_x, L_y] = i\hbar L_z$. The generalized uncertainty principle thus gives $\Delta L_x \Delta L_y \geq \hbar|\langle L_z \rangle|/2$. For $|l, 0\rangle$, we get $\hbar^2 l(l+1)/2 \geq 0$; this state is thus very far from a minimum uncertainty state. This should make sense; per our discussion above, the angular momentum is smeared out over the entire $x - y$ plane. We could get a smaller uncertainty for the same $\langle L_z \rangle$ by considering an eigenstate of $\hat{L}_x$ or $\hat{L}_y$. For $|l, l\rangle$, we the uncertainty principle gives $\hbar^2 l/2 \geq \hbar^2 l/2$. We see that for this case the inequality is saturated and we have a minimum uncertainty state.