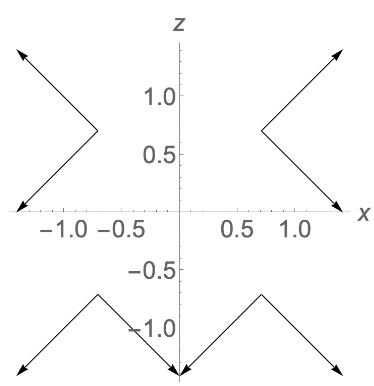
1. In three dimensions, we require
$$\int |\Psi_{3\mathrm{d}}|^2 dx\, dy\, dz = 1$$
as our normalization condition. The right-hand-side is dimensionless, so the left-hand-side must be as well. Each of dx, dy, dz have units of L, so $|\Psi_{3\mathrm{d}}|^2$ must have units of L^{-3}. The wavefunction itself thus has units of $L^{-3/2}$ in three dimensions.

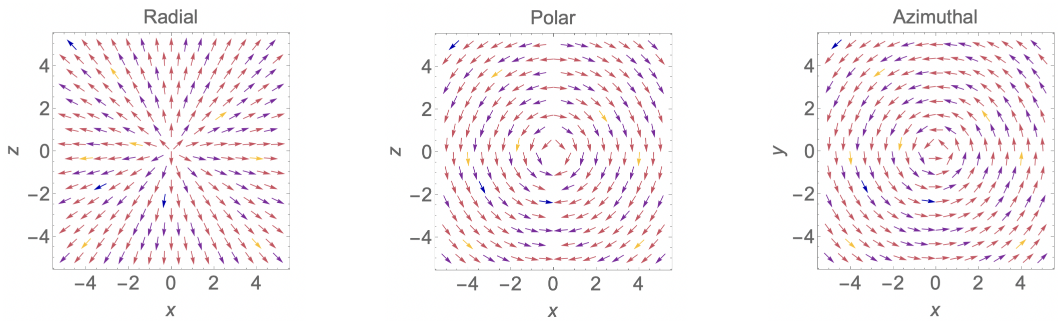
2. In one dimension, we require that
$$\int |\Psi(x)|^2 dx$$
be finite. Since
$$\int_{x_0}^{\infty} \frac{1}{x} dx = \log x|_{x_0}^{\infty}$$
is infinite and
$$\int_{x_0}^{\infty} \frac{1}{x^{1+p}} dx = -\frac{1}{px^p}\bigg|_{x_0}^{\infty} = \frac{1}{px_0^p}$$
is finite for $p > 0$, we have that the wavefunction must fall off faster than $1/\sqrt{x}$ as $x \to \infty$. (Note that the behavior for small x does not matter as long as the wavefunction is finite because the range of the integration is finite; it's only when going out to infinity that the wavefunction must be not only finite but also small enough.) In three dimensions, we now require that
$$\int |\Psi(x, y, z)|^2 dx\, dy\, dz = \int |\Psi(r)|^2 r^2 \sin\theta dr d\theta d\phi$$
be finite for a spherically symmetric wavefunction. The θ and ϕ integrals just give an overall factor of 4π; focusing on the r integral, we see there is an extra factor of r^2 as compared to the one-dimensional case. $|\Psi(r)|^2$ must thus decay faster than $1/r^3$ for the integral to infinity to be finite, so we need $\Psi(r)$ to fall off faster than $r^{-3/2}$ as $r \to \infty$.

3. (a) .



.
Those vectors pointing away from the origin are $\hat{\mathbf{r}}$; the other four are $\hat{\theta}$. The $\hat{\phi}$ point into the page on the right and out of the page on the left.

(b) .



.

The probability current densities are shown above for particular 2D slices. Per the continuity equation, having a probability density that is constant in time requires having a probability current with zero divergence. The purely radial current has a source at the origin, so it is not divergenceless; the purely polar current has a source on the positive z-axis and a sink on the negative z-axis, so it is not divergenceless. The purely azimuthal current has no sources or sinks, so it is the only one of the three consistent with a constant probability density.

(c) Looking at the expression for the probability current, we see that it is proportional to the difference of the two terms $\Psi^* \vec{\nabla} \Psi$ and $\Psi \vec{\nabla} \Psi^*$. These terms are complex conjugates of each other, so their difference is proportional to the imaginary part of one of them. Furthermore, since each term has one factor of Ψ and one of Ψ^*, any overall phase will cancel. We thus only have a nonzero imaginary part if the gradient generates an imaginary part. From the expressions in lecture, we know that the only non-real part of $\psi_{n,l,m}$ comes from the ϕ dependence in the form of $e^{im\phi}$. A ϕ derivative will bring down a factor of im, which has a factor of i. This means the only possible non-zero spherical component of $\vec{J}$ is the ϕ component.

(d) From the previous part, we know we only need to calculate the ϕ component. Since the only ϕ dependence of $\psi_{n,l,m}$ is $e^{im\phi}$, the effect of a ϕ derivative is to bring down a factor of im. We thus have

$$\begin{aligned}
\vec{J} &= -\frac{i\hbar}{2M}\left(e^{iE_n t/\hbar}\psi^*_{n,l,m}(\vec{r})(im)e^{-iE_n t/\hbar}\psi_{n,l,m}(\vec{r})\hat{\phi} - e^{-iE_n t/\hbar}\psi_{n,l,m}(\vec{r})(-im)e^{iE_n t/\hbar}\psi^*_{n,l,m}(\vec{r})\hat{\phi}\right) \\
&= \frac{\hbar m}{M}\left|\psi_{n,l,m}(\vec{r})\right|^2\hat{\phi}
\end{aligned}$$