Physics 115B, Mastery Questions for Section 10 Spring 22

Today we will be exploring discrete symmetries.

1. In this problem, we are going to derive some interesting properties of time reversal. We will denote parity inversion by the Π operator and time reversal by the Θ operator.

   (a) How do the position and momentum operators transform under parity? Specifically, what are $\Pi^{-1}x\Pi$ and $\Pi^{-1}p\Pi$?

   (b) Now consider the canonical commutation relation $[x,p] = i\hbar$. Compute $\Pi^{-1}[x,p]\Pi$ by explicitly writing out the commutator, inserting copies of the identity operator $1 = \Pi\Pi^{-1}$ where necessary, and using part (a). Compare with $\Pi^{-1}i\hbar\Pi$.

   (c) How does angular momentum transform under parity? This transformation means angular momentum is an *axial vector* (also called a *pseudovector*), as opposed to a *polar vector* (generally just called a vector) like linear momentum.

   (d) We can also consider the discrete symmetry of time-reversal, denoted by Θ, which exchanges t for $-t$. How should the position and momentum operators transform under time reversal? (Classical intuition gives the correct answer.)

   (e) Repeat part (b) for time reversal by explicitly computing $\Theta^{-1}[x,p]\Theta$ and comparing with $\Theta^{-1}i\hbar\Theta$. You should find that we require $\Theta^{-1}i\Theta = -i$. This is the antilinearity of time reversal: the action of Θ on scalars is complex conjugation.

   (f) For spin-0 wavefunctions, we have the property $\Theta\psi(x,t) = \psi(x,t)^*$. (Note that this does indeed flip the time-dependence, which schematically looks like $e^{-iEt/\hbar}$ and goes to $e^{+iEt/\hbar}$ under complex conjugation.) What is Θ^2? (The result turns out to be true for all bosons, not just those with spin-0.)

   (g) Classically, how does angular momentum transform under time reversal?

   (h) This is *almost* correct in quantum mechanics, but not the full story. It turns out that, for a spin-1/2 particle,

   $$\Theta\left[a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle\right] = a^*\Theta\left|\uparrow\right\rangle + b^*\Theta\left|\downarrow\right\rangle = a^*\left|\downarrow\right\rangle - b^*\left|\uparrow\right\rangle.$$

   Note that time-reversal does send up to down and down to up, but in addition there's the purely quantum-mechanical effect of the terms gaining a relative sign (which could not have been predicted on the basis of classical mechanics). What is the result of applying Θ^2? (The result turns out to be true for all fermions, not just those with spin-1/2.)

2. In this problem, we are going to analyze discrete symmetries in the context of classical electromagnetism. Recall the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

   (a) Noting how velocity and acceleration transform under parity, how must $\mathbf{E}$ and $\mathbf{B}$ transform under parity in order for the force law to remain valid?

   (b) Noting how velocity and acceleration transform under time reversal, how must $\mathbf{E}$ and $\mathbf{B}$ transform under time reversal in order for the force law to remain valid?

   (c) Define *charge conjugation* as the operation that flips the sign of all charges (this is only a classical version of the actual definition, which only exists in quantum field theory.) Consider Gauss's law in differential form,

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0.$$

   How must $\mathbf{E}$ transform under charge conjugation in order for Gauss's law to remain valid?

   (d) Consider Faraday's law in differential form,

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t.$$

   How must $\mathbf{B}$ transform under charge conjugation in order for Faraday's law to remain valid?

   (e) Using parts (a)-(d), how do $\mathbf{E}$ and $\mathbf{B}$ transform under the combined action of all three operations? (cf. CPT theorem)