Physics 115B, Mastery Questions for Section 1 Spring 22

We've now made the leap to quantum mechanics in three dimensions. Today our goal will be to build some intuition for what that implies.

(1) In one dimension, you could figure out that the wavefunction $\Psi_{\rm 1d}$ has units of $L^{-1/2}$ using the normalization condition $\int |\Psi_{\rm 1d}|^2 dx = 1$ – the RHS is dimensionless, so the LHS must be as well. By analogous reasoning, what are the units of the wavefunction $\Psi_{\rm 3d}$ for a particle in three dimensions?

(2) Many sensible statements in one dimension (normalizability, the conservation of probability, hermiticity of various operators, etc.) required the wavefunction for physical states to fall off faster than $1/\sqrt{x}$ as $x \to \infty$. What is the equivalent requirement in three dimensions? For simplicity, consider a spherically symmetric wavefunction $\Psi(r)$ in three dimensions, and consider how it must fall off in r as $r \to \infty$.

(3) The probability current for a particle of mass M moving in three dimensions and described by a state $\Psi(\vec{r}, t)$ is

$$\vec{J} = -\frac{i\hbar}{2M}\left(\Psi^* \vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right)$$

As a reminder, in spherical coordinates,

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Just as in one dimension, the probability current $\vec{J}$ and the density $\rho = |\Psi|^2$ satisfy a continuity equation, which is the natural generalization of the one-dimensional case:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

This may be expressed in integral form using the divergence theorem as

$$\frac{dP}{dt} = -\oint_S \vec{J} \cdot \hat{n}\, dA$$

where P is the probability of finding the particle within a volume V bounded by the surface S.

(a) At a point in each of the 4 quadrants in the $x - z$ plane, sketch the directions of the radial, azimuthal, and polar unit vectors $\hat{r}, \hat{\theta}$, and $\hat{\phi}$.

(b) Consider the probability current density in spherical coordinates. Make qualitative sketches that depicy a purely radial probability current density, a purely azimuthal probability current density ($\hat{\phi}$ direction), and a purely polar probability current density ($\hat{\theta}$ direction). Which of these could be consistent with a probability density that is everywhere constant in time?

(c) Consider a particle of mass M that moves in a central potential $V(r)$. Imagine that it is in a stationary state $\Psi_{n,\ell,m}(\vec{r},t) = e^{-iE_nt/\hbar}\psi_{n,\ell,m}(\vec{r})$ of energy E_n. Which of the spherical components of $\vec{J}$ can be nonzero?

(d) Find an expression for $\vec{J}$ in terms of $|\psi_{n,\ell,m}(\vec{r})|^2$, explicitly working out the dependence on the quantum numbers n, ℓ, m.