Transformation $\quad U(\vec{\alpha}) = e^{-i\vec{\alpha}\vec{K}}$

$\vec{\alpha}$: real numbers

$\vec{K}$: set of operators $\quad$ GENERATORS OF TRANSF

$\underline{U^{+}U = 1}$ unitary $\quad U^{-1} = U^{+}$

$$U^{+}U = \left(1 + i\vec{\alpha}\vec{K}^{+} + \ldots\right)\left(1 - i\vec{\alpha}\vec{K} + \ldots\right)$$

$$U^{+}U = 1 + i\vec{\alpha}\underbrace{(K^{+} - K)}_{\text{must be } = 0 \text{ for } U^{+}U = 1} + \ldots$$

$K^{+} = K$, K hermitian , observable

Examples translations $\vec{k} = \vec{P}$ (give or take $\hbar$)

rotations $\vec{k} = \vec{J}$

Isospin $\vec{k} = \frac{\vec{\sigma}}{2}$ $\vec{\sigma}$ = Pauli matrices

Symmetry $\boxed{U^\dagger H U = H}$

Operators transform $\hat{O} \to U^\dagger \hat{O} U$

① TRANSITIONS

$a \rightarrow b$, $\quad A_{ab} = \langle b | \hat{O} | a \rangle$

$\hat{O}$ = some operator contains details of interaction
function of H

$|A_{ab}|^2$ = measure of the strength of transition

For decays, lifetime $\propto \dfrac{1}{|A_{ab}|^2}$

For scattering, cross section $\propto |A_{ab}|^2$

$$|a'\rangle = U|a\rangle \qquad |b'\rangle = U|b\rangle$$

$$A_{ab} = \langle b|\hat{O}|a\rangle = \langle b|U^\dagger U \hat{O} U^\dagger U|a\rangle$$
$$= \langle b'|U\hat{O}U^\dagger|a'\rangle$$

if U good sym $\Rightarrow$ $\hat{O} \to U^\dagger \hat{O} U = U \hat{O} U^\dagger = \hat{O}$

$$A_{ab} = \langle b'|\hat{O}|a'\rangle = A_{a'b'}$$

SAME AMPLITUDE FOR TRANSITION BETWEEN STATES RELATED BY "GOOD" SYMM. OPERATION

② $[K_i, H] = 0$ if U is good symm

Consider infinitesimal $U = 1 - i\alpha_i K_i$

H invariant $\rightarrow$ $U^\dagger H U = H$

$$\left(1 + i\alpha_i K_i^\dagger\right) H \left(1 - i\alpha_i K_i\right) = H$$

To first order in α

$$H + i\alpha_i \left(K_i^\dagger H - H K_i\right) = H$$

But $K_i^\dagger = K_i \Rightarrow [K_i, H] = 0$

(3) Conservation laws

$$K_i|a\rangle = k_a|a\rangle \qquad \text{if } |a\rangle \text{ eigenstate of } K_i$$
$$K_i|b\rangle = k_b|b\rangle \qquad \text{if } |b\rangle \text{ eigenstate of } K_i$$

$$[K_i, \hat{O}] = 0$$

$$\langle b|[K_i, \hat{O}]|a\rangle = 0$$

$$\langle b|K_i^{\dagger}\hat{O} - \hat{O}K_i|a\rangle = 0 \qquad K_i^{\dagger} = K_i$$

$$(k_b - k_A)\langle b|\hat{O}|a\rangle = 0$$

either $\langle b|\hat{O}|a\rangle = A_{ab} = 0$ OR $\underline{k_A = k_b}$

④ DEGENERACY

Take $|a\rangle$ to be eigenstate of H

$$H|a\rangle = E_a|a\rangle$$

Take $|b\rangle \neq |a\rangle$, to be eigenstate of H

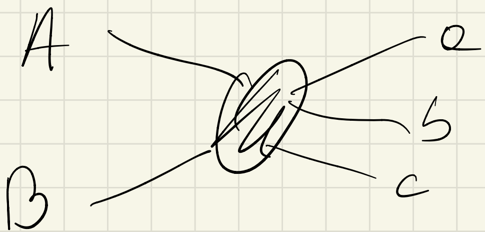
$$H|b\rangle = E_b|b\rangle$$

Assume $|b\rangle = U|a\rangle \leftarrow$

$$E_a = \langle a|H|a\rangle = \underbrace{\langle a|U^\dagger}_{\langle b|} H \underbrace{U|a\rangle}_{|b\rangle} = \langle b|H|b\rangle = E_b$$

(5) Continuous Sym $\Rightarrow$ Additive Cons Laws

example: momentum



$AB \to abc$

single particle states of definite momenta

initial state $|i\rangle = |\vec{p}_A\rangle |\vec{p}_B\rangle$

final state $|f\rangle = |\vec{p}_a\rangle |\vec{p}_b\rangle |\vec{p}_c\rangle$

Assume translational invariance

$$U(\vec{\alpha})\,|i\rangle = e^{-i\vec{\alpha}\cdot\hat{\vec{P}}_A}|\vec{p}_A\rangle \; e^{-i\vec{\alpha}\cdot\hat{\vec{P}}_B}|\vec{p}_B\rangle$$

$\vec{P}_A$ = momentum of A $\qquad \hat{\vec{P}}_A$ = momentum operator of A

$$e^{-i\vec{\alpha}\hat{\vec{P}}_A}|\vec{P}_A\rangle = e^{-i\alpha\vec{P}_A}|\vec{P}_A\rangle$$

$$\longrightarrow U(\vec{\alpha})|i\rangle = e^{-i\alpha(\vec{P}_A+\vec{P}_B)}|\vec{P}_A\rangle|\vec{P}_B\rangle = e^{-i\vec{P}_{INIT}}|i\rangle$$

$$U(\vec{\alpha})|f\rangle = e^{-i\alpha\vec{P}_{FINAL}}|f\rangle$$

$$\langle f|\hat{O}|i\rangle = \langle f|U^{+}\hat{O}U|i\rangle =$$

$$\langle f|\hat{O}|i\rangle = \underline{e^{+i(\vec{P}_F-\vec{P}_i)}}\langle f|\hat{O}|i\rangle$$

either $\langle f|\hat{O}|i\rangle = 0$ $\underline{\underline{OR}}$ $\boxed{\vec{P}_F = \vec{P}_I}$

⑥ DISCRETE SYM $\Rightarrow$ MULTIPLICATIVE CONS. LAW

Can I write parity $P = e^{i\vec{\alpha}\cdot\vec{K}}$ NO

Because P is discrete symm, not parametrized by a continous quantity

eg rotations parametrized by angles

$$P|a\rangle = P_a|a\rangle \qquad P^2|a\rangle = |a\rangle$$

$$P^2|a\rangle = P_a P|a\rangle = P_a^2|a\rangle \Rightarrow P_a = \pm 1$$

$$\langle b|\hat{O}|a\rangle = \langle b|\underbrace{P^\dagger}\,O\,\underbrace{P|a\rangle} = P_a P_b \langle b|\hat{O}|a\rangle$$

$$P_a \cdot P_b = 1 \quad \text{or} \quad \langle b|\hat{O}|a\rangle = 0$$

$P_a = P_b = 1$

$P_a = P_b = -1$

GO BACK TO ISOSPIN

proton = nucleon with Isospin up
neutron = nucleon with isospin down

$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Good sym $\begin{pmatrix} a \\ b \end{pmatrix} \Longrightarrow U \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad U =$ "rotation" in Isospin space

if we do not consider trivial $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\alpha}$

Then $U = e^{-i\vec{\alpha}\vec{I}}$ $\quad \vec{I} = \frac{1}{2}\vec{\sigma}$ $\quad \vec{\sigma}$ = Pauli Matrices

In general rotation $R = e^{-i\vec{\alpha}\vec{J}/\hbar}$ ✓

and in the case of $j = 1/2$ (ang mom $= 1/2$) $j = s = 1/2$

$R = e^{-i\vec{\alpha}\vec{S}/\hbar}$ $\qquad$ $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

- Ang mom consrved $\longrightarrow$ Isospin conserved

if rotational invariance

if only strong interaction is important

- for ang mom we choose bon's states that are simultaneous eigenstates of J^2, J_3
for isospin, choose eigenstates of I^2 and I_3

- In strong interaction processe I^2, I_3 conserved

- Matrix elements indep of I_3

We now understand isospin a bit better

The fundamental isospin doublet is made up of up/down quarks

$$\begin{pmatrix} u \\ d \end{pmatrix} \qquad Q(u) = \tfrac{2}{3}e \qquad Q(d) = -\tfrac{1}{3}e$$

almost same mass, and same strong interactions

$$p = uud \qquad n = udd$$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \oplus 0) \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$$

$I = 3/2$ states with 3 quarks exist

uuu	Δ^{++}	$\vert 3/2 \ 3/2 \rangle$
uud	Δ^{+}	$\vert 3/2 \ 1/2 \rangle$
udd	Δ^{0}	$\vert 3/2 \ -1/2 \rangle$
ddd	Δ^{-}	$\vert 3/2 \ -3/2 \rangle$

Antiquarks

$$\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \begin{array}{l} \leftarrow I = 1/2 \quad I_3 = +1/2 \\ \leftarrow I = \frac{1}{2} \quad I_3 = -1/2 \end{array}$$

$u\bar{d}$ $\quad |1\ 1\rangle = \pi^+$ | $\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$ $|1\ 0\rangle$ π^0 | $\bar{u}d = \pi^-$

$+\frac{1}{2}$ $+\frac{1}{2}$

A third quark — strange s

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to U \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Approx sym because strange quark is noticeably heavier

$m(u) \sim 2\ MeV/c^2$

$m(d) \sim 5\ MeV/c^2$

$m(s) \sim 96\ MeV/c^2$

Triump of theory

Predicted sss Ω^-

$\frac{1}{2} \otimes \frac{1}{2}$

	$\uparrow\uparrow$	$\uparrow\downarrow$	$\downarrow\uparrow$	$\downarrow\downarrow$
S_3	$\underline{1}$	0	0	-1

$\uparrow\uparrow \Rightarrow I_3 = 1$

$(\hat{S}_1 + \hat{S}_2)^2 \uparrow\uparrow = \hat{S}_1^2 \uparrow\uparrow + \hat{S}_2^2 \uparrow\uparrow + 2\vec{S}_1\vec{S}_2 \uparrow\uparrow$

$L_3|l\,m\rangle = m\hbar|l\,m\rangle$

$S^2\uparrow = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)\uparrow$

$2\hbar^2 \uparrow\uparrow = \hbar^2(1+1)1\uparrow\uparrow$

$\uparrow\uparrow \quad I = 1, \; I_3 = 1 = |1\;1\rangle$

$L^2|l\,m\rangle = l(l+1)\hbar^2$

$S_- \uparrow\uparrow \to |1\;0\rangle = \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}$

$|l\,m\rangle$

$|0\;0\rangle = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}$