Selection Rules

some operator

Matrix element $A = \langle n'\ell'm'|X|n\ell m\rangle$

Suppose X has some well defined parity properties

$$X \to \Pi^\dagger X \Pi = \lambda X \qquad \lambda = \pm 1$$

$$A \to \underbrace{\langle n'\ell'm'|\Pi^\dagger}_{(-1)^{\ell'}} X \underbrace{\Pi|n\ell m\rangle}_{(-1)^{\ell}} = \langle n'\ell'm'|\lambda X|n\ell m\rangle$$

$$(-1)^{\ell'}(-1)^{\ell}\langle n'\ell'm'|X|n\ell m\rangle = \lambda\langle n'\ell'm'|X|n\ell m\rangle$$

$$= \lambda(-1)^{\ell+\ell'}\langle n'\ell'm'|X|n\ell m\rangle = \underbrace{\langle n'\ell'm'|X|n\ell m\rangle}_{A}$$

$$\underline{A} \to \cancel{\langle n'l'm' | X | nlm \rangle} = \underbrace{(-1)^{l+l'}}\lambda A$$

$$\nsim$$

Start over

$$A \to \langle n'l'm' | \underbrace{\Pi^{+}}_{(-1)^{l'}} X \underbrace{\Pi}_{(-1)^{l}} | nlm \rangle = \lambda A$$

$$A \to (-1)^{l+l'} A = \lambda A$$

$$\lambda = (-1)^{l+l'}$$

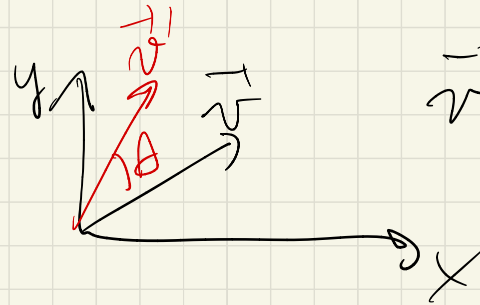
if $\lambda = 1$ $l+l'$ even

if $\lambda = -1$ $l+l'$ odd

Dipole operator $\vec{d} = q\vec{r}$

Rotations

2D



$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Rotate about $\hat{z}$ by θ

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R(\theta) \text{ around } z \text{ axis}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

In 3D

θ_x about $\hat{x}$ -
$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}$$

θ_y about $\hat{y}$ $\quad R_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$

θ_z about $\hat{z}$ $\quad R_z(\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Ouuuuu

$R = R_x R_y R_z$ order matters!!

Make $\theta_x \, \theta_y \, \theta_z$ very small $\quad \cos\theta_i \simeq 1 - \cancel{\frac{\theta^2}{2}} \quad \sin\theta \simeq \theta$

$R \to 1 + \vec{\theta}\vec{\mathcal{L}}$ $\quad \vec{\theta} = (\theta_x \; \theta_y \; \theta_z)$ all θ_i small

$\vec{\mathcal{L}} = (\mathcal{L}_x \; \mathcal{L}_y \; \mathcal{L}_z)$ $\quad \mathcal{L}_z = \begin{pmatrix} 1 & -\theta_z & 0 \\ \theta_z & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ etc

Remember for translations 1D

For small a $T(a) = 1 - ia\hat{k}$ $\hat{k}$ = operator)

For large a $T(a) = \exp(-ia\hat{k}) = \exp\left(-\frac{ia\hat{p}}{\hbar}\right)$

$R(\vec{\theta}) = 1 - i\vec{\theta}\vec{k}$ $\quad -i\vec{k} = \vec{L}$ $\quad \vec{k} = +i\vec{L}$

definition of $\vec{k}$

Finite $\vec{\theta}$

$$R(\vec{\theta}) = \exp(-i\vec{\theta}\vec{k})$$

$$= 1 - i\vec{\theta}\vec{k} + \frac{(-i\vec{\theta}\vec{k})^2}{2!} + \cdots$$

$$[k_i\ k_j] = i\,\varepsilon_{ijk}\,k_k \qquad [L_i\ L_j] = i\hbar\,\varepsilon_{ijk}\,L_k$$

This is suggestive that

$$R(\vec{\theta}) = \exp\left(-i\,\frac{\vec{\theta}\,\vec{L}}{\hbar}\right)$$

Let's justify this suggeustion

Under rotation $\vec{r} \to \vec{r}\,' = R\,\vec{r}$

For quantum states $|\psi\rangle \to |\psi'\rangle$

For wavefunctions $\psi'(\vec{r}\,') = \psi(\vec{r})$

This must be true!

$|\psi(\vec{r})|^2 = |\psi'(\vec{r}')|^2$ becouse the prob distribution somewhere in space after rotation must be same as the corresponding position

$\vec{r}' = R\vec{r}$

$$\psi'(\vec{r}') = \psi(\vec{r})$$

$$\psi'(\vec{r}') = \psi(R^{-1}\vec{r}')$$

relabel

$$\psi'(\vec{r}) = \psi(R^{-1}\vec{r})$$

how wavefunction changes

Rotation by θ around $\hat{z}$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

If θ small $\quad R(\theta) \simeq \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R^{-1}(\theta) = \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The new wavefunction after $R(\theta)$ is

$$\psi(R^{-1}\vec{r}) = \psi(x + y\theta \quad y - x\theta \quad z)$$

$$\psi(R^{-1}\vec{r}) = \psi(x, y, z) + y\theta \frac{\partial \psi}{\partial x} - x\theta \frac{\partial \psi}{\partial y}$$

(Taylor expansion)

$$\psi(R^{-1}\vec{r}) = \psi'(\vec{r}) - i\theta \underbrace{\left(iy\frac{\partial}{\partial x} - ix\frac{\partial}{\partial y} \right)}_{\frac{1}{\hbar}L_z} \psi$$

$$\psi'(\vec{r}) = \underbrace{\left(1 - \frac{i\theta L_z}{\hbar} \right)}_{R(\theta)} \psi(\vec{r})$$

Transformation of wavefunction for small rotation by θ around $\hat{z}$

If I make θ finite

$$R(\theta) = \exp\left(-\frac{i\theta L_z}{\hbar} \right)$$

$$\psi' = R(\theta)\psi$$

ISOSPIN SYMMETRY

Nuclear Physics 1930s

p & n are ~ same as far as strong interaction is concerned $m(p) \approx m(n)$

Good approx sym in nuclear physics

interchange $p \longleftrightarrow n$

Consider p & n as single entities of "nucleon"

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Invariant unvariant

(Physics) $\begin{pmatrix} p \\ n \end{pmatrix} \Longrightarrow U \begin{pmatrix} p \\ n \end{pmatrix}$

$U = 2\times 2$ unitary $UU^{\dagger} = 1$

4 complex numbers = 8 real numbers

Unitarity $U^{\dagger}U = 1 \Rightarrow 4$ constraints

$8 - 4 = 4$ real parameters for U

One of possible parameters is trivial

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\alpha} \qquad \begin{pmatrix} p \\ n \end{pmatrix} \to U \begin{pmatrix} p \\ n \end{pmatrix}$$

Get rid of this type of rotation – U is parametrized by 3 real parameters

Can show that U can be written as

$$U = e^{i\vec{\alpha}\vec{I}} \quad \text{or} \quad e^{-i\vec{\beta}\vec{I}} \qquad \vec{\alpha} = -\vec{\beta}$$

$$\vec{\alpha} = (\alpha_1\ \alpha_2\ \alpha_3)$$

$$\vec{I} = \frac{1}{2}\vec{\sigma}$$

Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ etc $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$U = e^{i\vec{\alpha}\vec{I}}$$ $\vec{I}$ are generators of rotations in $\binom{p}{n}$ space

$$\vec{I} = \vec{\sigma}$$

Compare with $R(\theta) = e^{-i\vec{\theta}\vec{L}/\hbar}$

For spin $\frac{1}{2}$ representation $R = e^{-i\vec{\theta}\vec{S}/\hbar}$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \Rightarrow R = e^{-i\frac{\vec{\theta}\vec{\sigma}}{2}}$$

Isospin I $\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} I_3 = +1/2 \\ I_3 = -1/2 \end{pmatrix}$