PARITY   P or π

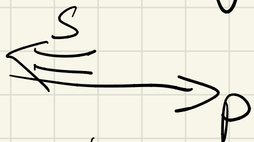
Parity $\vec{r} \to -\vec{r}$

4 fundamental interaction
- gravity
- e&m
- strong

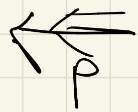
] Parity good

- weak] parity not good

Neutrinos only interact weakly

ν 

Left handed for interaction ✓

Parity Transform 

does not interact ✓

$$\vec{L} = \vec{r} \times \vec{p}$$

under parity
$$\vec{r} \to -\vec{r}$$
$$\vec{p} \to -\vec{p}$$
$$\vec{L} \to \vec{L}$$

1D for now

$$\Pi|x\rangle = |-x\rangle$$

$$\Pi|\psi\rangle = |\phi\rangle$$

$$\psi(x) \xrightarrow{\Pi} \psi(-x)$$

$$\Pi^2\psi(x) = \Pi(\Pi\psi(x)) = \Pi\psi(-x) = \psi(x)$$

$\Pi^2 = I$ obvious

$$\Pi\Pi^{-1} = I \Rightarrow \boxed{\Pi^{-1} = \Pi}$$

Consider $\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\psi(x)$

change of variables $y = -x$

$dx = -dy$ $\quad -\infty \to \infty$ for x

$\quad +\infty \to -\infty$ for y

$$\langle\psi|\psi\rangle = \int_{+\infty}^{-\infty} (-dy)\, \psi^*(-y)\, \psi(-y)$$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} (+dy)\, \psi^*(-y)\, \psi(-y)$$

Call $y = x$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(-x)\, \psi(-x)$$

$$\Pi\psi(x) = \psi(-x)$$

$$\left(\Pi\psi(x)\right)^* = \psi^*(-x)$$

$$\Rightarrow \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \left(\Pi\psi(x)\right)^* \Pi\psi(x)\, dx$$

$$\langle\psi|\psi\rangle = \langle\psi|\Pi^\dagger\Pi|\psi\rangle$$

$\Pi^\dagger\Pi = 1$ unitory operator

We had $\Pi^2 = 1$ $\Pi^{-1} = \Pi$

$\Pi^{-1} = \Pi^{\dagger}$

$\Pi = \Pi^{\dagger}$ Π Hermitian

Π is associated with an observable

Parity is an observable

Eigenvalues of parity λ ?

$$\Pi \psi(x) = \lambda \psi(x)$$

$$\Pi^2 \psi(x) = \lambda \Pi \psi(x) = \lambda \psi(x)$$

Identity

$$\psi(x) = \lambda^2 \psi(x)$$

$$\lambda = \pm 1$$

$\lambda = 1$ "even parity"

$\lambda = -1$ "odd parity"

eigenfunct.

$$\underbrace{\Pi \, \psi(x)}_{= \psi(-x)} = \pm \psi(x)$$

$$\psi(x) = \pm \psi(-x)$$

even or odd eigenfunctions

Homework: show $\hat{x} \to \Pi^\dagger \hat{x} \, \Pi \to -\hat{x}$

$\hat{p} \to \Pi^\dagger \hat{p} \, \Pi \to -\hat{p}$

Generically if $\hat{O}$ is an operator and T is some transformation

$$\hat{O} \longrightarrow T^\dagger O \, T$$

(showed last week)

$H \Rightarrow \Pi^{+} H \Pi$ transformation of H

$$H \Rightarrow H = \Pi^{+} H \Pi$$

$$\Rightarrow [H\ \Pi] = 0$$

In 1D $\quad H = \frac{p^2}{2m} + V(x)$

p^2 invariant under parity

$V(x) = V(-x)$ for $[H\ \Pi] = 0$

We can then find a complete set of simultaneous eigenstates of Π AND of H

eigenfunctions of Π are $\psi(x) = \pm \psi(-x)$

Then eigenfunctions of H can be expressed as a set of even and odd eigenfunctions "vectors" that are odd under parity such as $\vec{r}$, $\vec{p}$ are called VECTORS

those that are even under parity such as $\vec{L}$, $\vec{S}$ are called PSEUDOVECTORS

HW

Spherical Potentials

$$V(\vec{r}) = V(r) = V(-\vec{r})$$

$V(r)$ is invariant under parity

So, if H contains $V(r)$, the eigenfunction can be written as even or odd under parity

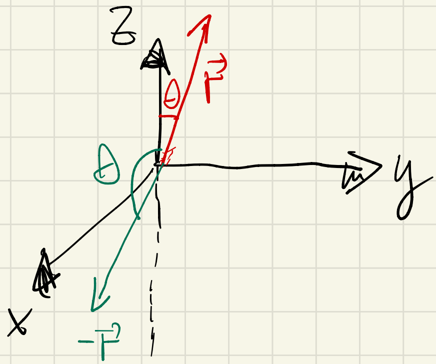
$$H\psi(\vec{r}) = E\psi(\vec{r}) \quad \text{with } V(r) \text{ (central)}$$

$$\psi(\vec{r}) = \phi(r)\, Y_\ell^m(\theta, \phi) \Longleftarrow$$

Under parity

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \rightarrow -x \\ y &= r\sin\theta\sin\phi \rightarrow -y \\ z &= r\cos\theta \longrightarrow -z \end{aligned}}$$

$\theta \rightarrow ?$ $\quad \phi \rightarrow ?$ $\quad r \rightarrow r$

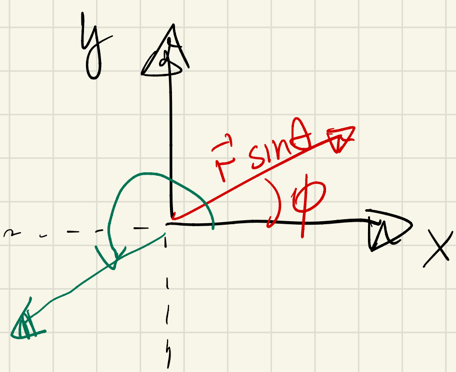


θ is angle wrt $\hat{z}$

$0 \le \theta \le \pi$

$\theta \rightarrow \pi - \theta$

$$\boxed{\begin{aligned} \cos\theta &\rightarrow -\cos\theta \\ \sin\theta &\rightarrow \sin\theta \end{aligned}}$$



$\phi \rightarrow \pi + \phi$

$$\boxed{\begin{aligned} \sin\phi &\rightarrow -\sin\phi \\ \cos\phi &\rightarrow -\cos\phi \end{aligned}}$$

What happens to $Y_\ell^m(\theta,\phi)$ under parity

$$\theta \to \pi - \theta \qquad \phi \to \pi + \phi$$

$$Y_\ell^m(\theta,\phi) = C\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

so under parity

$$Y_\ell^m(\theta,\phi) \to C\, P_\ell^m(-\cos\theta)\, e^{im\phi}\, e^{im\pi}$$

$$e^{im\pi} = \cos m\pi + i\,\cancel{\sin m\pi} = (-1)^m$$

$$Y_\ell^m(\theta,\phi) \to C\,(-1)^m\, P_\ell^m(-\cos\theta)\, e^{im\phi}$$

$$P_\ell^m(x) = \underbrace{(-1)^m (1-x^2)^{m/2}}_{\text{stays same as } x\to -x} \underbrace{\left(\frac{d}{dx}\right)^m}_{(-1)^m \text{ as } x \to -x} \underbrace{P_\ell(x)}_{?} \leftarrow$$

$$P_\ell(x) = \sum_{n=0}^{\ell} a_n x^n$$

if ℓ is even only $a_0, a_2, \ldots a_\ell \neq 0$

If ℓ is odd on $a_1, a_3 \ldots a_\ell \neq 0$

$$P_0(x) = c_0 \qquad P_1(x) = c_1 x$$

$$P_2(x) = c_2 x^2 + c_0'$$

Under $x \to -x$ $\quad P_\ell(x) = (-1)^\ell P_\ell(-x)$

$$P_\ell^m(x) = (-1)^{\ell+m} P_\ell^m(-x)$$

We had

$$Y_\ell^m(\theta, \phi) \to C(-1)^m P_\ell^m(-\cos\theta) e^{im\phi}$$

$$\to C(-1)^m (-1)^{\ell+m} P_\ell^m(\cos\theta) e^{im\phi}$$

$$(-1)^{\ell+2m} = (-1)^\ell (-1)^{2m} = (-1)^\ell$$

$$\boxed{Y_\ell^m(\theta, \phi) = (-1)^\ell Y_\ell^m(\pi - \theta, \phi + \pi)}$$

Spherical Harmonics:

even $\ell \rightarrow$ even parity

odd $\ell \rightarrow$ odd parity