**Parity** \( P \) or \( \Pi \)

Parity \( P \rightarrow -P \)

4 fundamental interaction \[ \begin{array}{c}
\text{gravity} \\
\text{e.m.} \\
\text{strong} \\
\text{weak} \end{array} \]

Parity good

Parity not good

Neutrinos only interact weakly

\( \nu \overset{\text{S}}{\leftrightarrow} \bar{\nu} \rightarrow L \leftarrow P \) Left handed for interaction \( \nu \)

Parity Transform \( \nu \overset{\text{S}}{\leftrightarrow} \bar{\nu} \leftarrow P \)

\( \bar{L} = P \times \bar{P} \)

under parity \[ \begin{array}{c}
\bar{P} \rightarrow -\bar{P} \\
\bar{P} \rightarrow -\bar{P} \\
L \rightarrow \bar{L} \end{array} \]
1D for now

\[ \Pi |x\rangle = | -x \rangle \]
\[ \Pi |\psi\rangle = | \phi \rangle \]
\[ \Psi(x) \xrightarrow{\Pi} \Psi(-x) \]
\[ \Pi^2 \Psi(x) = \Pi (\Pi \Psi(x)) = \Pi \Psi(-x) = \Psi(x) \]

\[ \Pi^2 = I \]

obvious

\[ \Pi \Pi^{-1} = I \implies \Pi^{-1} = \Pi \]

Consider \[ \langle \psi | \psi \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \psi(x) \]

change of variables \[ y = -x \]

\[ dx = -dy \]
\[ -\infty \to \infty \quad \text{for} \quad x \]
\[ +\infty \to -\infty \quad \text{for} \quad y \]
\[ \langle \psi | \bar{\psi} \rangle = \int_{-\infty}^{+\infty} (-dy) \psi^*(-y) \psi(-y) \]
\[ \langle \bar{\psi} | \psi \rangle = \int_{-\infty}^{+\infty} (+dy) \psi^*(-y) \psi(-y) \]

Call \( y = x \)

\[ \langle \phi | \bar{\psi} \rangle = \int_{-\infty}^{+\infty} d\pi \psi^*(-\pi) \psi(-\pi) \]

\[ \Pi \psi(x) = \psi(-x) \]
\[ (\Pi \psi(x))^* = \psi^*(-x) \]

\[ \langle \psi | \Pi \psi \rangle = \int_{-\infty}^{+\infty} (\Pi \psi(x))^* \Pi \psi(x) \, dx \]

\[ \langle \psi | \bar{\psi} \rangle = \langle \psi | \Pi^+ \Pi \Pi^+ \Pi \psi \rangle \]

\[ \Pi^+ \Pi = 1 \]

Unitary operator
We have $\Pi^2 = 1$, $\Pi^{-1} = \Pi$.

$\Pi$ is Hermitian $\Pi = \Pi^\dagger$.

$\Pi$ is associated with an observable.

parity is an observable.

Eigenvalues of parity $\lambda$?

$\Pi \psi(x) = \lambda \psi(x)$

$\Pi^2 \psi(x) = \lambda (\Pi \psi(x))$

Identity

$\psi(x) = \lambda^2 \psi(x)$

$\lambda = \pm 1$.
\[ \lambda = 1 \quad " \text{even parity} " \]
\[ \lambda = -1 \quad " \text{odd parity} " \]

\[ \Pi \psi(x) = \pm \psi(x) \]
\[ \psi(x) = \pm \psi(-x) \]

"even or odd eigenfunctions"

Homework: show \[ \hat{\theta} \rightarrow \Pi^+ \hat{\theta} \Pi \rightarrow - \hat{\theta} \]
\[ \hat{\rho} \rightarrow \Pi^+ \hat{\rho} \Pi \rightarrow - \hat{\rho} \]

Generically if \( \hat{\theta} \) is an operator and \( T \) is some transformation
\[ \hat{\theta} \rightarrow T^+ \hat{\theta} T \quad (\text{showed last week}) \]
$H = \Pi^+ H \Pi$ transformation of $H$

$H \Rightarrow H = \Pi^+ H \Pi$

$\Rightarrow [H, \Pi] = 0$

In 1D: $H = \frac{p^2}{2m} + V(x)$

$p^2$ invariant under parity

$V(x) = V(-x)$ for $[H, \Pi] = 0$

We can then find a set of simultaneous eigenstates of $\Pi$ AND of $H$

Eigenfunctions of $\Pi$ are $\Psi(x) = \pm \Psi(-x)$

Then eigenfunctions of $H$ can be expressed as a set of even and odd eigenfunctions.
"vectors" that are odd under parity such as $\bar{F}$, $\bar{p}$ are called VECTORS

those that are even under parity such as $\bar{I}$, $\bar{s}$ are called PSEUDO VECTORS

\[
\begin{align*}
\mathbf{M}
\end{align*}
\]

Spherical Potentials

\[
 V(\vec{r}) = V(r) = V(-\vec{r})
\]

$V(r)$ is invariant under parity

So, if $H$ contains $V(r)$, the eigenfunctions can be written as even or odd under parity

\[
 H \psi(\vec{r}) = E \psi(\vec{r}) \text{ with } V(r) \text{ (center)}
\]
\[ \Psi(r) = \Phi(r) \, Y_n^m(\theta, \phi) \]

Under parity

\[ x = r \sin \theta \cos \phi \rightarrow -x \]
\[ y = r \sin \theta \sin \phi \rightarrow -y \]
\[ z = r \cos \theta \rightarrow z \]

\[ \theta \rightarrow \pi - \theta \]
\[ \cos \theta \rightarrow - \cos \theta \]
\[ \sin \theta \rightarrow \sin \theta \]

\[ \phi \rightarrow \pi + \phi \]
\[ \sin \phi \rightarrow - \sin \phi \]
\[ \cos \phi \rightarrow - \cos \phi \]

\[ \theta \text{ is angle wrt } z \]
\[ 0 \leq \theta \leq \pi \]
What happens to \( Y_e^m(\theta, \phi) \) under parity

\[ \theta \rightarrow \pi - \theta, \quad \phi \rightarrow \pi + \phi \]

\[ Y_e^m(\theta, \phi) = C \, P_e^m(\cos \theta) \, e^{im\phi} \]

so under parity

\[ Y_e^m(\theta + \phi) \rightarrow C \, P_e^m(-\cos \theta) \, e^{im\phi} \, e^{im\pi} \]

\[ e^{im\pi} = \cos m\pi + i \sin m\pi = (-1)^m \]

\[ Y_e^m(\theta, \phi) \rightarrow C \, (-1)^m \, P_e^m(-\cos \theta) \, e^{im\phi} \]

\[ P_e^m(x) = (-1)^m \, (1 - x^2)^m \frac{d^m}{dx^m} P_e(x) \]

\[ P_e(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{if } l \text{ is even} \]

\[ \text{only } a_0, a_2, \ldots, a_l \neq 0 \]
If \( l \) is odd on \( \theta_1, \theta_2, \ldots, \theta_l \neq 0 \)

\[
P_0(x) = c_0, \quad P_1(x) = c_1 x
\]

\[
P_2(x) = c_2 x^2 + c_1
\]

Under \( x \to -x \) \( P_e(x) = (-1)^l P_e(-x) \)

\[
P_e^m(x) = (-1)^{l+m} P_e^m(-x)
\]

We had

\[
Y^m_e(\theta, \phi) \to C (-1)^m P_e^m(-\cos \theta) e^{i m \phi}
\]

\[
\to C (-1)^m (-1)^{l+m} P_e^m(\cos \theta) e^{i m \phi}
\]

\[
(-1)^{l+2m} = (-1)^l (-1)^{2m} = (-1)^l
\]

\[
Y^m_e(\theta, \phi) = (-1)^l Y^m_e(\pi - \theta, \phi + \pi)
\]
Spherical Harmonics:

Even $\ell \rightarrow$ Even parity
Odd $\ell \rightarrow$ Odd parity