Since $T(a+b) = T(a)T(b)$
build $T(a)$ by summing many small (infinitesimal) translations

$T(a)$ take a to be small

$$T(a) \simeq T(0) + a \left.\frac{dT(a)}{da}\right|_{a=0}$$

$T(0) = 1$ operator

$$T(a) = 1 + a \left.\frac{dT(a)}{da}\right|_{a=0}$$

operator

$$f(x) = f(0) + x \left.\frac{df}{dx}\right|_{x=0}$$

number

function

$$\hat{k} = +i \left.\frac{dT(a)}{da}\right|_{a=0} \qquad \left.\frac{dT(a)}{da}\right|_{a=0} = -i\hat{k}$$

$$T(a) = 1 - ia\hat{k}$$

$$T(-a) = 1 + ia\hat{k} = T(a)^{-1}$$

Last lecture $T(a)^{-1} = T(a)^{\dagger} = 1 + ia\hat{k}$

$$T(a)^{\dagger} = (1 - ia\hat{k})^{\dagger} = 1 + ia\hat{k}^{\dagger}$$

$\hat{k}$ is Hermitian

$$\frac{dT(a)}{da} = \lim_{\varepsilon \to 0} \frac{T(a+\varepsilon) - T(a)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{T(a)T(\varepsilon) - T(a)}{\varepsilon}$$

$$= \left(\lim_{\varepsilon \to 0} \left[\frac{T(0+\varepsilon) - T(0)}{\varepsilon} \right] \right) T(a)$$

The limit is $\left.\frac{dT(a)}{da}\right|_{a=0} = -i\hat{k}$

$$\frac{dT(a)}{da} = -i\hat{k}\,T(a)$$

$$T(a) = A e^{-ia\hat{k}}$$

When $a=0$ $\quad T(a=0) = \underline{1} \Rightarrow \underline{\underline{A=1}}$

$$T(a) = e^{-ia\hat{k}} = 1 - ia\hat{k} - \frac{1}{2}a^2\hat{k}^2 + \dots$$

$$\hat{k}\psi = i\left(\frac{dT(a)}{da}\psi\right)\Bigg|_{a=0} \qquad T(a)\psi(x) = \psi(x-a)$$

$$\hat{k}\psi = i\frac{d}{da}\psi(x-a)\Bigg|_{a=0}$$

what is $\frac{d}{da}\psi(x-a)\Big|_{a=0}$??

$y = x - a$ $\qquad \frac{d}{da} = -\frac{d}{dy}$

$y = x - a$

$a \to 0$

$y \to x$

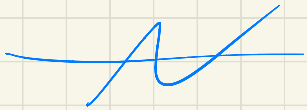
$$= -\frac{d}{dy}\psi(y)\Big|_{y=x} = -\frac{d}{dx}\psi(x)$$

$$\hat{K}\psi = +i\frac{d}{da}\psi(x-a)\Big|_{a=0} = -i\frac{d}{dx}\psi$$

$$\hat{K} = -i\frac{d}{dx} \Rightarrow \hat{k} = \frac{\hat{p}}{\hbar}$$

$$T(a) = \exp\left(-\frac{i a \hat{p}}{\hbar}\right)$$

$\frac{\hat{p}}{\hbar}$ = <u>Generator</u> of translations



So far we looked $T(a)\psi(x) = \psi(x-a)$

Some other operator $\hat{O}$

want its expectation value ...

$$\langle T\psi|\hat{O}|T\psi\rangle = \langle\psi|T^\dagger \hat{O} T|\psi\rangle$$

keep state as it was and

"transform" operator $\hat{O} \to T^\dagger \hat{O} T$

Imagine that H is invariant under symm $T^\dagger H T = H$

$$\underbrace{T T^\dagger}_{=1} H T = T H$$

$$HT = TH \qquad [H\ T] = 0$$

$$H = \frac{\hat{p}}{2m} + V(\vec{r})$$

If I translate by $\vec{a}$, $V(\vec{r}) \to V(\vec{r}+\vec{a})$

For H to be invariant under translations by any $\vec{a}$, $V = \text{constant}$

CM

$V = \text{constant}$

1D $\quad F = ma = m\frac{dp}{dt} \qquad F = -\frac{dV}{dx}$

$V = \text{const}, \ F = 0 \Rightarrow \frac{dp}{dt} = 0$

$\Rightarrow p$ constant

$[H, T(a)] = 0$ for any a

$T(a) = \exp\left(-\frac{i a \hat{p}}{\hbar}\right)$

Take a to be very small

$T(a) = 1 - \frac{i a \hat{p}}{\hbar}$

If $[H\ \hat{O}]$ and $\frac{\partial \hat{O}}{\partial t} = 0 \Rightarrow \langle O \rangle = \text{const}$

$$\frac{\partial}{\partial t}\langle O\rangle = \frac{\partial}{\partial t}\langle \psi|\hat{O}|\psi\rangle$$

$$= \langle \frac{\partial}{\partial t}\psi|\hat{O}|\psi\rangle + \langle\psi|\cancel{\frac{\partial}{\partial t}\hat{O}}|\psi\rangle + \langle\psi|\hat{O}|\frac{\partial}{\partial t}\psi\rangle$$

$= 0$

$$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \qquad \frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}H|\psi\rangle$$

$$\frac{\partial}{\partial t}\langle O\rangle = \frac{1}{\hbar}\langle -iH\psi|\hat{O}|\psi\rangle + \frac{1}{\hbar}\langle\psi|\hat{O}|-iH\psi\rangle$$

$$= \frac{1}{\hbar}\langle\psi|+iH^{+}\hat{O}|\psi\rangle + \frac{1}{\hbar}\langle\psi|-iH\hat{O}|\psi\rangle$$

$$\frac{\partial}{\partial t}\langle O\rangle = \frac{i}{\hbar}\langle\psi|\underbrace{H\hat{O}-\hat{O}H}_{[H\ \hat{O}]}|\psi\rangle$$

if $[H\ \hat{o}] = 0$ then $\boxed{\frac{\partial}{\partial t}\langle o\rangle = 0}$

We had $[\hat{P}\ H] = 0$

$\Rightarrow \frac{d\langle p\rangle}{dt} = 0$

$$[H, T(a)] = \cancel{[H, 1]} - \frac{ia}{\hbar}[H, \hat{P}] = 0$$

$$= 0$$

$$\Rightarrow [H, \hat{P}] = 0$$

H commutes with generator of translations

Take $\hat{O}$, with discrete eigenvalues

λ_n, eigenstates $|\phi_n\rangle$

If I am in state $|\psi\rangle$

Prob of measuring λ_n

$$P(\lambda_n) = |\langle\phi_n|\cancel{\hat{O}}|\psi\rangle|^2$$

$$|\psi\rangle = |\psi(t)\rangle = \sum c_m e^{-\frac{iE_m t}{\hbar}} |\psi_m\rangle$$

time independent

$$p(\lambda_n) = \left|\sum_m \underline{c_m} e^{-iE_m t/\hbar} \langle\phi_n|\cancel{\hat{O}}|\psi_m\rangle\right|^2$$

If $[H\ \hat{O}] = 0$ then

eigenstates of $\hat{O}$ and H can be made to be the same

$$|\phi_n\rangle = |\psi_n\rangle$$

eigenstates of O

eigenstates of H

$$\langle\phi_n|\hat{O}|\psi_m\rangle = \lambda_n \langle\phi_n|\psi_m\rangle$$

$$\hat{O}|\phi_n\rangle = \lambda_n|\phi_n\rangle$$

$$= \lambda_n \langle\psi_n|\psi_m\rangle$$

$$= \delta_{nm} \times \lambda_n$$

$$P(\lambda_n) = \left|\sum_m c_m e^{iE_m t/\hbar} \langle\phi_n|\psi_m\rangle\right|^2$$

$$= \langle\psi_n|\psi_m\rangle$$

$$= \delta_{nm}$$

$$\boxed{P(\lambda_n) = |c_n|^2}$$

indep of time