DIRAC NOTATION, bras & kets

We have used $\Psi(x)$ ie wavefunctions to represent state of QM system

We have also written $|\Psi\rangle$

More abstract

Also used for spin states $|\uparrow\rangle$

There is also correspondence $\Psi(x)$

$$\Psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x)\, e^{-ipx/\hbar}\, dx$$

position space $\longleftrightarrow$ momentum space

"change of representation"

Finite vector space

$$\vec{a} = a_i \hat{e}_i = \sum a_i \hat{e}_i = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\vec{a} = a'_i \hat{e}'_i = \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix}$$

$|a\rangle$ Abstract way of representing state

"ket"

eigenvectors of $\hat{x}|x\rangle = x|x\rangle$

eigenvectors of $\hat{p}|p\rangle = p|p\rangle$

eigenvector of $\hat{S}_z|\uparrow\rangle = +\frac{1}{2}\hbar|\uparrow\rangle$

~~N~~

Inner product of two vectors

$|a\rangle, |b\rangle$ $\langle b|a\rangle$

$\langle b|a\rangle$ What is $\langle b|$ "bra"

$|a\rangle$ is "ket" represents a state

Bra is not a vector

One to one correspondence $|b\rangle \leftrightarrow \langle b|$

Mathematically speaking:

- ket lives in a vector space V
- bras are elements of a dual space of linear functionals on V

linear functionals

$\langle b|$ acts on $|a\rangle \to$ number

$\langle b|a\rangle$ = complex number

Operator on ket $\longrightarrow$ another ket

$$\hat{O}|a\rangle = |c\rangle$$

Functional on ket $\longrightarrow$ number

$$\langle b|a\rangle = \text{number}$$

Finite $\checkmark$

$$|a\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad |b\rangle = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\langle b|a\rangle$

$$\langle b| = (b_1^* - - - - - b_n^*)$$

$$\langle b|a\rangle = (b_1^* - - - b_n^*)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} =$$

$$\langle b|a\rangle = b_i^* a_i \qquad \langle b| = (|b\rangle)^\dagger$$

OUTER PRODUCT

$|a\rangle\langle b|$???

finite V

$$|a\rangle\langle b| = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1^* \ldots b_n) = \begin{pmatrix} N \times N \\ \text{matrix} \end{pmatrix}$$

$|a\rangle\langle b| \Rightarrow$ is an operator

$$(|a\rangle\langle b|)\,|c\rangle = |a\rangle \underbrace{\langle b|c\rangle}_{\text{number}} = \text{number}\ |a\rangle$$

Let's denote basis $|\phi_i\rangle$ orthonormal

$$|\psi\rangle = c_i |\phi_i\rangle$$

$$\langle\phi_j|\psi\rangle = c_i \langle\phi_j|\phi_i\rangle = c_i \delta_{ij} = c_j$$

$$\rightarrow |\psi\rangle = \underbrace{\langle\phi_i|\psi\rangle}_{=c_i} |\phi_i\rangle$$

$$|\psi\rangle = \sum_i \underbrace{|\phi_i\rangle\langle\phi_i|}_{\text{operator}}\psi\rangle$$

$$|\psi\rangle = \underbrace{\left(\sum_i |\phi_i\rangle\langle\phi_i|\right)}_{=\text{Identity}} |\psi\rangle$$

$$\sum_i |\phi_i\rangle\langle\phi_i| = \underline{1}$$

~~continou~~

For a continous set $|\phi(i)\rangle$

$$\int |\phi(i)\rangle\langle\phi(i)|\, di = \underline{1}$$

i is whatever labels the states, eg p or x or ....

$|\psi\rangle$ to $\psi(x)$

$$\psi(x) \equiv \langle x|\psi\rangle$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'| \; |\psi\rangle$$

$$\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx' \, \underbrace{\langle x|x'\rangle}_{\delta(x-x')} \langle x'|\psi\rangle$$

$$\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx' \underbrace{\langle x|x'\rangle}_{\delta(x-x')} \underbrace{\langle x'|\psi\rangle}_{\text{number}}$$

$$\langle x|\psi\rangle = \psi(x)$$

$$\psi(x) = \int_{-\infty}^{\infty} dx' \, \delta(x-x') \, \psi(x')$$

$\bar{x}$ = expectation value of x

$$\bar{x} = \int_{-\infty}^{\infty} x \, |\psi(x)|^2 \, dx \quad x$$

$$\bar{x} = \langle \psi | \hat{x} | \psi \rangle$$

$$\psi(x) = \langle x|\psi\rangle$$

$$\bar{x} = \int dx' \langle \psi | \underbrace{\hat{x} \, |x'\rangle}_{x'|x'\rangle} \langle x'| \, |\psi\rangle$$

$$x$$

$$\bar{x} = \int dx' \, x' \langle \psi|x'\rangle \langle x'|\psi\rangle = \int dx' \, x' \, |\langle x'|\psi\rangle|^2$$

Symmetries, Conservation laws

Spatial Translation (1D)

Classically: move particle by a

QM: take state and shift it by a in space

$T(a)$ = operator which shifts state by a

$$T(a)|x\rangle = |x+a\rangle$$

$$T(a)|\psi\rangle = |\phi\rangle$$

$$\psi(x) = \langle x|\psi\rangle$$

$$\phi(x) = \langle x|\phi\rangle = \langle x|T(a)|\psi\rangle$$

$$\phi(x) = \langle x | T(a) | \psi \rangle$$

$$\phi(x) = \int dx' \underbrace{\langle x | T(a) | x' \rangle}_{|x'+a\rangle} \underbrace{\langle x' | \psi \rangle}_{\psi(x')}$$

$$\phi(x) = \int dx' \underbrace{\langle x | x'+a \rangle}_{\delta(x-x'-a)} \psi(x')$$

$\rightarrow x' = x - a$

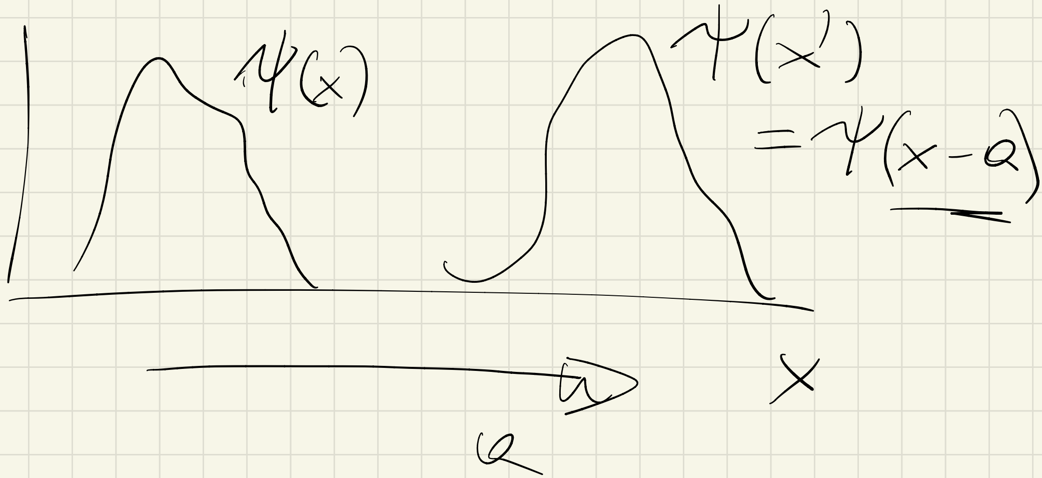
$$\boxed{\phi(x) = \psi(x-a)}$$

$$\boxed{T(a)|x\rangle = |x+a\rangle}$$

$$\psi(x) \rightarrow \psi(x-a)$$

$$|x\rangle \to |x+a\rangle$$

$$\psi(x) \to \psi(x-a)$$



$$T(0) = \mathbb{1}$$

$$T(a)\,T(b) = T(a+b)$$

$$T(a)^{-1} = T(-a)$$

$$T(a)^{-1} = T(a)^{\dagger}$$ unitary

$$\langle \phi | \phi \rangle = 1 \qquad \langle \psi | \psi \rangle = 1$$

$$\phi = T(a)\psi$$

$$\langle T(a)\psi \,|\, T(a)\psi \rangle = 1$$

$$\langle \psi | \underbrace{T^{\dagger}(a)\,T(a)}_{=1} | \psi \rangle = 1$$

$T(a)$ not hermitian

$$\boxed{T(a) = e^{-\frac{ia\hat{P}}{\hbar}}}$$