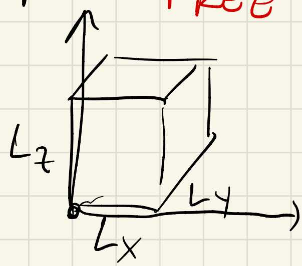
Recap

Simple model of solids: some electrons in outer shell are so loosely bound that we pretend they are free

FREE ELECTRON GAS



particle in a box

$V=0$ inside $V=\infty$ outside

$$\psi(\vec{r}) = A \sin k_x x \sin k_y y \sin k_z z$$

$$k_x = \frac{\pi n_x}{L_x} \qquad k_y = \frac{\pi n_y}{L_y} \qquad k_z = \frac{\pi n_z}{L_z}$$

$\vec{k} = (k_x \; k_y \; k_z)$ WAVE VECTOR

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2m}\left[\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right]$$

Compare with free particle

$$\psi = e^{i\vec{k}\vec{r}} = \cos \vec{k}\vec{r} + i \sin \vec{k}\vec{r}$$

$kx + ky + kz$

$$\vec{k} = \frac{\vec{p}}{\hbar} \qquad E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

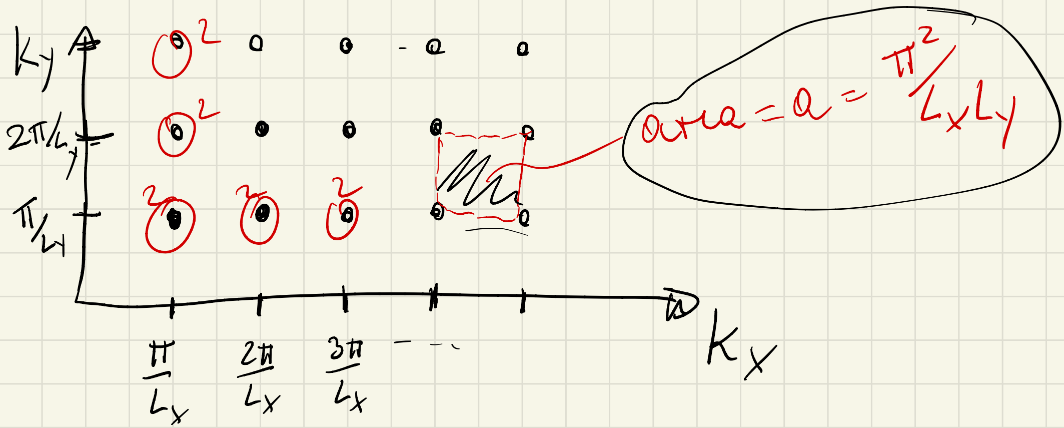
$\vec{k}$ can take any

For single electron lowest energy state

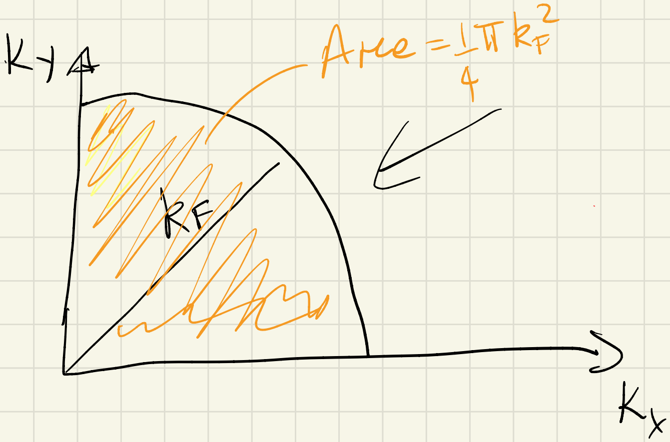
$n_x = n_y = n_z = 1$

Pauli exclusion principle (for half-integer spin $=$ fermions) says that fermions cannot be in the same quantum state

Because of 2 spin states $\uparrow$ or $\downarrow$, I can put 2 electrons in each "position" state. In 2D:



Grid is filled with $\sim 6\ 10^{23} - 10^{24}$ electrons



$E \propto k^2 = k_x^2 + k_y^2$

Fill the grid so as to make k^2 as small as possible

k_F = maximum of k

All states within k_F semicircle are filled

For a given $N = \#$ of free electrons we should be able to figure out k_F

$$N = \frac{\frac{1}{4}\pi k_F^2}{a} \times 2$$

from spin

Area of quadrant

Area occupied by one state

$$N = \frac{1}{2}\pi k_F^2 \Big/ \pi / L_x L_y$$

In 3D, Areas become volume

$$a = \frac{\pi^2}{L_x L_y} \rightarrow v = \frac{\pi^3}{L_x L_y L_z} = \frac{\pi^3}{V}$$

V = volume of solid

The quarter circle becomes eigth of sphere

$$\frac{1}{4}\pi k_F^2 \longrightarrow \frac{1}{8}\,\frac{4}{3}\pi k_F^3$$

" " " " algebra ....

$$N = \frac{V}{3}\,\frac{k_F^3}{\pi^2} \qquad \rho = \frac{N}{V} = \#\text{ of electrons}/\text{unit volume}$$

$$\boxed{k_F = \sqrt[3]{3\rho\pi}}$$

FERMI SURFACE = Boundary in k-space between the occupied and unoccupied states

FERMI ENERGY $E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$

TOTAL ENERGY OF ELECTRON GAS

I know energy at a given k

$$E = \frac{\hbar^2 k^2}{2m}$$

① We figure out how much energy is in shell $k \to k+dk$

dE

② this energy $dE \propto dk$

③ $\int dE = \int_0^{k_F} \underline{\quad}\, dk$

① $dE =$ Number of states in shell $\times$ Energy of each state

② $\dfrac{2 \cdot \text{Volume of shell}}{v}$ $= \dfrac{\frac{1}{8} 4\pi k^2 dk}{\pi^3 / V}$

$$dE = \frac{\hbar^2 V}{2m\pi^2} k^4 \, dk$$

$$E = \int_0^{k_F} \frac{\hbar^2 V}{2m\pi^2} k^4 \, dk = \frac{\hbar^2 V}{10 m \pi^2} k_F^5$$

$$E = \frac{\hbar^2 V}{10 m \pi^2} (3\rho\pi)^{5/3} \qquad \rho = \frac{N}{V}$$

$$E = \frac{\hbar^2}{10 m \pi^2} (3\pi N)^{5/3} V^{-2/3} = A V^{-2/3}$$

If volume increases energy decreases

$$\frac{dE}{dV} = -\frac{2}{3} A V^{-5/3} = -\frac{2}{3} \frac{A V^{-3/2}}{V} = -\frac{2}{3} \frac{E}{V}$$

$$\boxed{dE = -\frac{2}{3} E \frac{dV}{V}}$$

Loss of E by gas = work done on outside dW

$$dW = p\,dV = |dE| = \frac{2}{3} E \frac{dV}{V}$$

$$\boxed{P = \frac{2}{3} \frac{E}{V}}$$ Electron degeneracy pressure

Prevents collapse of a white dwarf!

p, n $S = 1/2$

$\checkmark$ ℓ

$|\psi\rangle \qquad \psi(x)$

$$\boxed{\langle x|\psi\rangle = \psi(x)}$$