# TIME TRANSLATIONS

1D $\quad H\psi(x,t) = i\hbar \frac{\partial\psi}{\partial t}$

$$\psi(x,t) = \psi(x,t=0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n\psi}{\partial t^n}\Big|_{t=0} t^n$$

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar} H\psi \qquad \frac{\partial\psi}{\partial t}\Big|_{t=0} = -\frac{i}{\hbar} H\psi(x,t=0)$$

1st term ↑

2nd term $$\frac{\partial^2\psi}{\partial t^2} = \underline{\frac{\partial}{\partial t}\left(-\frac{i}{\hbar}H\psi\right)} = -\frac{i}{\hbar}H\frac{\partial\psi}{\partial t} \cancel{-\frac{i}{\hbar}\left(\frac{\partial H}{\partial t}\right)\psi}$$

Assume $\frac{\partial H}{\partial t} = 0$

$$\underline{\underline{\frac{\partial^2\psi}{\partial t^2}}} = -\frac{i}{\hbar}H\frac{\partial\psi}{\partial t} = \underline{\underline{\left(-\frac{i}{\hbar}\right)^2}}\,\underline{\underline{H^2}}\,\underline{\psi}$$

$$\boxed{\frac{\partial^n\psi}{\partial t^n} = \left(-\frac{i}{\hbar}\right)^n H^n\psi}$$

$$\Psi(x,t) = \underbrace{\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{iH}{\hbar}\right)^n t^n\right]}_{U(t)=\exp\left(-\frac{iHt}{\hbar}\right)} \Psi(x, t=0)$$

$$\boxed{\Psi(x,t) = U(t)\,\Psi(x, t=0)}$$

$U(t)$ = time translation operator

Hamiltonian is generator of time translations

# Heisenberg Picture (HP)

So far: Schrödinger Picture (SP)

HP is an equivalent way to formulate QM

SP

$$H\,\Psi_s(x,t) = i\hbar \frac{\partial}{\partial t}\Psi_s(x,t)$$

In SP, $\Psi$ depends on $t$

$H$ is built up from $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ $\quad \hat{x} = x$

no explicit time dependence

In HP, wavefunction is independent of time

$$\psi_H(x) = \psi_S(x, t=0)$$

the time dependence goes in operators

$\hat{O}$ = operator in SP $(\hat{O}_S)$

$$\hat{O}_H = U^\dagger(t)\, \hat{O}_S\, U(t)$$

Physics cannot change SP $\to$ HP

In SP $\langle O \rangle = \int \psi_S^*(x,t)\, \hat{O}_S\, \psi_S(x,t)\, dx$

In HP $\langle O \rangle = \int \psi_H^*(x)\, \hat{O}_H\, \psi_H(x)\, dx$

$$\langle O \rangle = \int \psi_H^*(x)\, U^\dagger(t)\, \hat{O}_S\, \underline{U(t)\, \psi_H(x)}\, dx$$

$$U(t)\, \psi_H(x) = U(t)\, \psi_S(x, t=0) = \underline{\psi_S(x,t)}$$

$$\langle O \rangle = \int \psi_S^*(x,t)\, \hat{O}_S\, \psi_S(x,t)\, dx$$

$\langle O \rangle$ in HP $= \langle O \rangle$ in SP

$$O_H(t) = U^\dagger O_s U$$

$$\frac{dO_H(t)}{dt} = \frac{dU^\dagger}{dt} O_s U + U^\dagger O_s \frac{dU}{dt} + U^\dagger \frac{\partial O_s}{\partial t} U$$

$$U = \exp\left(-\frac{iHt}{\hbar}\right) \Rightarrow \frac{dU}{dt} = -\frac{iH}{\hbar} U(t)$$

$$\frac{dU^\dagger}{dt} = \left(\frac{dU}{dt}\right)^\dagger = \left(-\frac{iH}{\hbar} U\right)^\dagger = \frac{+i}{\hbar} U^\dagger H^\dagger = \frac{+i}{\hbar} U^\dagger H$$

$$\frac{dO_H}{dt} = \frac{i}{\hbar}\left(U^\dagger H^\dagger O_s U - U^\dagger O_s H U\right) + U^\dagger \frac{\partial O_s}{\partial t} U$$

$$[U\ H] = 0 \qquad [U^\dagger H] = 0$$

$$\frac{dO_H}{dt} = \frac{i}{\hbar}\Big(H \underbrace{U^\dagger O_S U}_{O_H} - \underbrace{U^\dagger O_S U}_{O_H} H\Big) + U^\dagger \frac{\partial O_S}{\partial t} U$$

HO - OH

[H O]

$$\boxed{i\hbar \frac{\partial O_H}{\partial t} = [O_H\ H] + i\hbar\, U^\dagger \frac{\partial O_S}{\partial t} U}$$

Classical Mechanics $\quad f(q, p, t)$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\{f\ H\} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}$$

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Take $O_s = \hat{X} = x$ what is $\hat{X}_H$ ?

$$\hat{X}_H = U^\dagger(t)\,\hat{X}_s\,U(t) = \underline{U^\dagger(t)\,x\,U(t)}$$

$$i\hbar \frac{d\hat{X}_H}{dt} = [\hat{X}_H\ H] + 0$$

$$i\hbar \frac{d\hat{X}_H}{dt} = [U^\dagger x U\ \ H] = U^\dagger\,[x\ \ H]\,U$$

I skipped steps $[U \; H] = 0$ $\qquad H = \dfrac{\hat{p}^2}{2m} + V(x)$

$$i\hbar \frac{d\hat{x}_H}{dt} = U^\dagger \left[ x \;\; \frac{\hat{p}^2}{2m} \right] U$$

because $[x \;\; V(x)]$ commute

$$i\hbar \frac{\partial \hat{x}_H}{dt} = \frac{1}{2m} U^\dagger \Big( \underbrace{[x \;\; \hat{p}]}_{i\hbar} \hat{p} + \hat{p} \underbrace{[x \;\; \hat{p}]}_{i\hbar} \Big) U$$

$$i\hbar \frac{\partial \hat{x}_H}{\partial t} = \frac{2i\hbar}{2m} \underbrace{U^\dagger \hat{p} \, U}_{\hat{P}_H}$$

$$\boxed{\frac{d\hat{x}_H}{dt} = \frac{1}{m} \hat{P}_H}$$

Similar algebra

$$\frac{d}{dt}\hat{P}_H = -\frac{dV_H}{dt}$$