ISOSPIN SYMMETRY

Nuclear physics, 1930's

$m_{\text{proton}} \approx m_{\text{neutron}}$

As far as strong interactions, they are same

Of course, very different for E&M - p is charged -

But strong interactions (eg in nuclei) is much

"Stronger" than E&M (or weak)

There is then a good symmetry that if we

interchange $p \leftrightarrow n$ nothing changes (so far

as strong interaction is concerned)

Heisenberg proposed that $n$ & $p$ could

be considered single entity of "NUCLEON"

$p = (1) \quad n = (0) \quad$ (does it look like spin?)

Physics invariant under $p \leftrightarrow n$ - But more

then that. Invariant under rotation of states

$(p) \rightarrow U(p)$
$U = 2 \times 2$ complex matrix

$\Rightarrow 4$ complex numbers $\Rightarrow 8$ real numbers

But it must be unitary $U^\dagger U = 1$

That's 4 constraints $\Rightarrow 8 - 4$ real parameters

One of the parameter is trivial

$U = (1, 0) e^{i\alpha}$

This $U$ is not really a rotation - Just adding a meaningless phase - Therefore 3 real parameters

It can be shown that $U$ can be written as

$e^{i\vec{\alpha} \cdot \vec{\sigma}}$ or $e^{-i\beta \vec{\sigma}}$ ($\vec{\alpha} = -\vec{\beta}$)

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ real vector

$\vec{\sigma} = \frac{1}{2} \vec{\sigma}$

$\vec{\sigma}$ Pauli matrices

$\vec{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\vec{\sigma}_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\vec{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
\( \vec{I} \) are the generators of rotations in isospin space.

Compare with rotation operator \( R = e^{\frac{i \vec{L} \cdot \vec{P}}{\hbar}} \)

For \( L = \frac{1}{2} \), i.e. spin \( S = \frac{1}{2} \)

\[ R = e^{-i \vec{A} \vec{S}} \quad \vec{S} = \frac{\hbar}{2} \vec{\sigma} \quad (1,1) \]

except for factors of \( \hbar \) \( \vec{S} \) and \( \vec{I} \) have the same matrix representation. Careful, they are not the same operator!

Therefore \( p \) and \( n \) behave just like spin up and spin down in their algebra.

Can compare \( p \) and \( n \) just like we combine spins to make states of \( I = 1,0 \) (two nucleons), \( I = \frac{3}{2} \) or \( \frac{1}{2} \) (three nucleons) etc.

Some Clebsch-Gordan coefficients

**BUT THERE IS MORE!**
If we have rotational symmetry:

- angular momentum is conserved ($J$)
- we conventionally choose basis states that are eigenstates of $J^2$ with eigenvalues $j(j+1)\hbar^2$ and $j$ integer or half integer, and eigenstates of $J_3$ with eigenvalues $j\hbar, (j+1)\hbar, \ldots, -(j/2)\hbar$

If we have isospin symmetry

- isospin is conserved ($I$)
- eigenstates of $I^2, I_3$ with some properties as above (except for factors $\hbar$)

The conservation of $I$ is realized in scattering processes, e.g. $AB \rightarrow CD$ or $AB \rightarrow CDE$, and in decay processes $A \rightarrow BC (D, E, \ldots)$

$J_z$ is also conserved (of course)

And the strength of the process (cross
section in case of scattering, inverse of lifetime in the case of decays) is independent of $I_3$

Why? because rotating in isospin space changes $I_3$, and this rotation is a symmetry

Now we understand isospin a bit better. The fundamental isospin doublet is the up/down quark doublet ($I = \frac{1}{2}$)

$\begin{align*}
(u) & \quad \text{They have different charges} \\
(d) & \quad Q(u) = \frac{2}{3} e \quad Q(d) = -\frac{1}{3} e,
\end{align*}$

almost same mass, and same strong interaction

If I combine 3 quarks

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = (1 + 0) \times \frac{1}{2} = \frac{3}{2} + \frac{1}{2} + \frac{1}{2}$$

I can get $I = \frac{3}{2}$ or $\frac{1}{2}$
The proton and neutron are the $I = \frac{1}{2}$ combination of $uud$ and $udd$. You would think that then there should also be 3-quark combinations with $I = \frac{3}{2}$. You would be right:

- \( \Delta^{++} \ uuu \quad I = \frac{3}{2} \quad I_3 = \frac{3}{2} \)
- \( \Delta^{+} \ uud \quad I = \frac{3}{2} \quad I_3 = \frac{1}{2} \)
- \( \Delta^{0} \ udd \quad I = \frac{3}{2} \quad I_3 = -\frac{1}{2} \)
- \( \Delta^{-} \ ddd \quad I = \frac{3}{2} \quad I_3 = -\frac{3}{2} \)

Other isospin multiplets exist by including the antiquark doublet \( (-\bar{u}) \)

(The minus sign is “techniscal”)

- \( \Pi^{+} \ u\bar{d} \quad I = 1 \quad I_3 = 1 \)
- \( \Pi^{0} \ (u\bar{u} - d\bar{d})/\sqrt{2} \quad I = 1 \quad I_3 = 0 \)
- \( \Pi^{-} \ \bar{u}d \quad I = 1 \quad I_3 = -1 \)
Remember: Isospin symmetry is not an exact symmetry even neglecting E&H because the masses of the u and d quark are not exactly the same—
isospin symmetry can be extended to a larger symmetry of rotations between 3 states, once we consider the strange quark

\[
\begin{pmatrix}
  u \\
  d \\
  s
\end{pmatrix}
\rightarrow
U
\begin{pmatrix}
  u \\
  d \\
  s
\end{pmatrix}
\]

An even more approximate symmetry since \(m(s) \gg m(u), m(d)\)

\[
\begin{align*}
m(u) &= 2 \text{ MeV/c}^2 \\
m(d) &= 5 \text{ MeV/c}^2 \\
m(s) &\approx 96 \text{ MeV/c}^2
\end{align*}
\]
Gell-Mann and Zweig were able to make sense of the plethora of particles that were being discovered in experiments, by postulating the existence of "quarks" as fundamental building blocks. (Although Gell-Mann did not think that they were real, at least initially - he thought that they were "mathematical objects")

The prediction and subsequent discovery of the \( \Xi^- \) particle (a \( 1555 \) state) was the triumph of the theory.